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Stability of the Steady States in Multidimensional Reaction Diffusion Systems Arising in Combustion Theory

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Abstract: We prove that the steady states of a class of multidimensional reaction–diffusion systems are asymptotically stable at the intersection of unweighted space and exponentially weighted Sobolev spaces, paying particular attention to a special case, namely, systems of equations that arise in combustion theory. The steady-state solutions considered here are the end states of the planar fronts associated with these systems. The present work can be seen as a complement to the previous results on the stability of multidimensional planar fronts.

Keywords: planar front; steady state; nonlinear stability; exponential weights

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1. Introduction

In this paper, we study the stability of the steady states of a class of reaction–diffusion systems associated with combustion problems in multidimensional cases. We study the following system of general reaction diffusion equations for $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$:

$$\mathbf{u}_t(t,\mathbf{x}) = D\Delta_{\mathbf{x}}\mathbf{u}(t,\mathbf{x}) + f(\mathbf{u}(t,\mathbf{x})), \mathbf{u} \in \mathbb{R}^n, d \geq 2, t \geq 0, n \geq 2, \tag{1}$$

where $\mathbf{u} = (u_1, u_2, ..., u_n)$ is the vector of the temperature and the substances taking part in the reactions, the diagonal matrix $D = \mathrm{diag}(d_1, d_2, ..., d_n)$ is the transport coefficient matrix (the coefficients of thermal diffusivity and diffusion coefficients), and the non-linear term $f(\mathbf{u}(t,\mathbf{x}))$ in the system is used to describe the reaction rate of each substance. We later provide several hypotheses on the nonlinear terms of this general system that allow us to apply the methods in this paper. A typical example is the following system:

$$\begin{cases} u_{1t}(t,\mathbf{x}) = \Delta_{\mathbf{x}}u_1(t,\mathbf{x}) + u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})),\ u_1, u_2 \in \mathbb{R}, \\ u_{2t}(t,\mathbf{x}) = \epsilon\Delta_{\mathbf{x}}u_2(t,\mathbf{x}) - \kappa u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})),\ \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{2}$$

where the reaction rate has an Arrhenius temperature dependence

$$g(u_1) = \begin{cases} e^{-\frac{1}{u_1}} & \text{if } u_1 > 0; \\ 0 & \text{if } u_1 \leq 0. \end{cases} \tag{3}$$

While we assume $d \geq 2$ in this paper, note that d usually has a definite physical meaning only when $d = 1, 2$, or 3 is adopted. Here, u_1 and u_2 denote the dimensionless temperature and concentration of the initial substance, respectively, ϵ is the ratio of the coefficient of diffusion to the coefficient of heat conduction, and κ is the stoichiometric coefficient, which satisfies $0 \leq \epsilon < 1$ and $\kappa > 0$. Chemical reaction waves turn one equilibrium state into another; here, we consider combustion waves that involve a strong temperature dependence in the reaction rate. Following the conventions of combustion theory, we assume that the chemical reaction rate is so small at low temperatures compared to the maximum temperature of the combustion wave that their rates can be assumed, to a fairly good approximation, to be equal to zero ([1]), which in the mathematical formulation means setting a cut-off, as in (3).

The class of systems (1) permits various types of traveling wave solutions. Traveling waves are waves that maintain a certain shape while propagating at a fixed speed in a medium; they are widely present in a variety of natural phenomena modeled by nonlinear evolutionary equations that often describe chemical or physical processes shifting from one equilibrium state to another. We consider a traveling wave solution moving in the direction of a given vector $\mathbf{e} \in \mathbb{R}^d$ with a constant speed $c > 0$, and without loss of generality, we assume that $\mathbf{e} = (1, 0, \ldots, 0)$. In (1), we employ a t-dependent change of variables $z = x_1 - ct, x_j = x_j, j = 2, \ldots, d$ and re-denote $\mathbf{x} = (z, x_2, \ldots, x_d)$ again. Then, system (1) in this moving coordinate system becomes

$$\mathbf{u}_t(t, \mathbf{x}) = D\Delta_{\mathbf{x}}\mathbf{u}(t, \mathbf{x}) + c(\mathbf{e} \cdot \nabla_{\mathbf{x}})\mathbf{u}(t, \mathbf{x}) + f(\mathbf{u}(t, \mathbf{x})). \quad (4)$$

It can be shown that each solution of (1) corresponds to a solution of (4), and vice versa.

In particular, we focus on the steady states of the wave fronts. By *wave fronts*, we mean solutions $\phi(z)$ of system (1) having limits $z \to \pm\infty$,

$$\lim_{z \to \pm\infty} \phi(z) = \mathbf{u}_\pm,$$

where $\mathbf{u}_+ \neq \mathbf{u}_-$, meaning that ϕ satisfies the ordinary differential equation

$$D\phi_{zz}(z) + c\phi_z(z) + f(\phi(z)) = 0.$$

Under circumstances of physical interest, these solutions approach both end states $\mathbf{u}_-$ and $\mathbf{u}_+$ at an exponential rate. The end states are usually not independent as such; a detailed discussion can be found in [2] (Chapter 8). We use Model (2) in Remark 1 as an example to illustrate how the states are chosen.

There have been a large number of papers devoted to the study of the existence and stability of combustion waves. In particular, when $d = 1$, we consider

$$\begin{cases} \phi_{1zz} + c\phi_{1z} + \phi_2 g(\phi_1) = 0, \\ \epsilon\phi_{2zz} + c\phi_{2z} - \kappa\phi_2 g(\phi_1) = 0, \end{cases}$$

with the end conditions $\phi_1(-\infty) = \phi_1^*$, $\phi_2(-\infty) = 0$, $\phi_1(+\infty) = 0$, and $\phi_2(+\infty) = 1$. Setting $p = \phi_{1z}$ and considering u_1 as an independent variable, we obtain

$$\begin{cases} \frac{dp}{d\phi_1} = -c - g(\phi_1)\frac{\phi_2}{p}, \\ \frac{d\phi_2}{d\phi_1} = \frac{-c}{\epsilon p}\left(\kappa(\phi_1 - \phi_1^*) + \phi_2\right) - \frac{\kappa}{\epsilon}, \end{cases}$$

with conditions $\phi_2(\phi_1^*) = 0$, $\phi_2(0) = 1$, and $p(\phi_1^*) = p(0) = 0$. With the aid of estimates of $p(\phi_1)$ from the above equations, the existence of solutions is established (cf. [3]). The wave speed c, as an important characteristic of combustion waves, is obtained mainly by approximate analysis and asymptotic methods, and in many cases analytical investigations are complemented by numerical studies.

In this paper, we focus on the stability of the steady states of this type of systems. In the physical and mathematical senses, instability can be understood as sensitivity to perturbations, i.e., the possibility that the propagation of a traveling wave is distorted or altered from the system state due to a perturbation, which eventually leads to abnormal appearance or abnormal steady-state output. A more precise definition can be found in [4]. B. Sandstede and A. Scheel [5], having analysed various instability mechanisms in reaction–diffusion systems, base their classification on the type of spectrum on the imaginary axis of the linear operator from linearization of the system. One particular case involves the essential instability [5] that arises when the essential spectrum (defined as consisting of all points on the spectrum that are not isolated eigenvalues of finite multiplicity (see [6], Chapter 5) crosses the imaginary axis. At this point, the proof of nonlinear stability is based upon the use of exponential weights for the essential spectrum (see, e.g., [7,8]) and on renormalization techniques to show that the nonlinear terms are asymptotically independent compared to linear diffusion.

The purpose of using an exponentially weighted space is that this usually allows one to shift the essential spectrum, which would otherwise cross the imaginary axis, to the left half of the complex plane, allowing the exponentially decaying properties of the associated semigroup to be used. An application of this approach to reaction diffusion equations can be found in a series of papers [9–13] which demonstrate the orbital stability of the traveling front by studying perturbations that are small in both unweighted and weighted Sobolev spaces. Subsequently, in [14], the authors established the existence of a stable foliation near the traveling front solution of the reaction diffusion system in one-dimensional space, i.e., the existence of a central manifold at each point on the front solution that attracts nearby solutions that are slightly perturbed to the front solution itself or to one of its translations. This result complements the orbit stability results in [11]. Then, in [15], the same authors extended the stability results for the traveling front solution of the reaction–diffusion system associated with the combustion model in [11] to a multidimensional case. However, the result in [15] was formulated under the assumption that the diffusion coefficients of the variables are identical, although this assumption often does not satisfy the characteristics of reaction–diffusion systems in reality.

2. Methods

To study the stability of $\phi(z)$, we can perturb the function ϕ by either:

(i) adding a function that depends only on one space variable z, that is, considering the solution $\mathbf{u}(t,\mathbf{x})$ of (4) with the initial condition

$$\mathbf{u}(0,\mathbf{x}) = \phi(\mathbf{x}\cdot\mathbf{e}) + \mathbf{v}(0,\mathbf{x}\cdot\mathbf{e})$$

with some $\mathbf{v} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^n$ with some $\mathbf{v}$ in the appropriate function space constructed later; or by

(ii) adding a function that depends on all spatial variables, that is, considering the solution $\mathbf{u}(t,\mathbf{x})$ of (4) with the initial condition

$$\mathbf{u}(0,\mathbf{x}) = \phi(\mathbf{x}\cdot\mathbf{e}) + \mathbf{v}(0,\mathbf{x})$$

with some $\mathbf{v} : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^n$ from an appropriate function space.

Note that under the first type of perturbation, the problem is indeed very similar to [11], except that the spatial variables are multidimensional. Thus, we focus here on the stability of the steady state of the front of system (4) under the second type of perturbation. The main advance of this paper compared to previous works is the extension of the result in [11] for a system of one-dimensional spatial variables to a system of multidimensional spatial variables, along with the absence of the assumption in [15] that the diffusion coefficients are the same for different system variables. Moreover, as we describe in the text, this type of equation has a special "product triangle" structure in the nonlinear reaction terms which is similar to the equations studied in the one-dimensional case from [10–12]; this class of nonlinear terms often appears in combustion models.

To better demonstrate how this "product triangle" structure can help to study the stability of the steady states, we begin with model case (2) for $\mathbf{u} = (u_1,u_2)^T \in \mathbb{R}^2$; in the moving coordinates $\mathbf{x} = (z, x_2, ..., x_d)$, this system becomes

$$\mathbf{u}_t(t,\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_{\mathbf{x}}\mathbf{u}(t,\mathbf{x}) + c(\mathbf{e}\cdot\nabla_{\mathbf{x}})\mathbf{u}(t,\mathbf{x}) + f(\mathbf{u}(t,\mathbf{x})), \tag{5}$$

where $f(\mathbf{u}(t,\mathbf{x})) = \begin{pmatrix} f_1(u_1,u_2) \\ f_2(u_1,u_2) \end{pmatrix} = \begin{pmatrix} u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})) \\ -\kappa u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})) \end{pmatrix}.$

According to the discussion presented earlier, a front solution $\phi = \phi(z)$ is t-independent and approaches the constant states $\mathbf{u}_+$ and $\mathbf{u}_-$ as $z \to \pm\infty$. System (5) has two types of steady state solutions: one when $u_1(\mathbf{x})$ equals a real constant and $u_2(\mathbf{x}) = 0$, and the other when $u_1(\mathbf{x}) = 0$ and $u_2(\mathbf{x})$ is equal to a real constant. In particular, we can choose $u_1 = 1/\kappa, u_2 = 0$, which is the equilibrium corresponding to the completely burned reactants located behind the front, and $u_1 = 0, u_2 = 1$, corresponding to the unburned substances. In other words, we choose $\mathbf{u}_- = (1/\kappa, 0)$ and $\mathbf{u}_+ = (0, 1)$. For a more detailed explanation of why $\mathbf{u}_-$ and $\mathbf{u}_+$ are chosen in this way, see [10] and Remark 1.

Here, we discuss in detail only the stability of $\mathbf{u}_-$, as the stability of $\mathbf{u}_+$ can be proven in precisely the same manner. We investigate perturbations of the state $\mathbf{u}_- = (1/\kappa, 0)$ that depend on all spatial variables of the system, that is, we consider the solutions $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_- + \mathbf{v}(t, \mathbf{x})$ of (5) with the initial conditions

$$\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_- + \mathbf{v}(0, \mathbf{x}),$$

where $\mathbf{v} = (v_1, v_2) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^2$ is taken from an appropriate function space. Substituting $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_- + \mathbf{v}(t, \mathbf{x})$ into system (5), we have

$$\mathbf{v}_t(t, x) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_{\mathbf{x}} \mathbf{v}(t, \mathbf{x}) + c\partial_z \mathbf{v}(t, \mathbf{x}) + f(\mathbf{u}_- + \mathbf{v}(t, \mathbf{x})). \tag{6}$$

Linearizing the nonlinearity $f(\mathbf{u}_- + \mathbf{v}(t, \mathbf{x}))$ at $\mathbf{u}_- = (1/\kappa, 0)$ provides

$$\begin{aligned} f(\mathbf{u}_- + \mathbf{v}(t, \mathbf{x})) &= f(\mathbf{u}_-) + \partial_{\mathbf{u}} f(\mathbf{u}_-)\mathbf{v}(t, \mathbf{x}) + H(\mathbf{v}(t, \mathbf{x})) \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \begin{pmatrix} v_1(t, \mathbf{x}) \\ v_2(t, \mathbf{x}) \end{pmatrix} + H(\mathbf{v}(t, \mathbf{x})), \end{aligned}$$

where we introduce the nonlinear term by

$$H(\mathbf{v}(t, \mathbf{x})) = f(\mathbf{u}_- + \mathbf{v}(t, \mathbf{x})) - f(\mathbf{u}_-) - \partial_{\mathbf{u}} f(\mathbf{u}_-)\mathbf{v}(t, \mathbf{x}). \tag{7}$$

Therefore, we have the following semi-linear equation for the perturbations of state $\mathbf{u}_-$:

$$\mathbf{v}_t(t, \mathbf{x}) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_{\mathbf{x}} \mathbf{v}(t, \mathbf{x}) + c\partial_z \mathbf{v}(t, \mathbf{x}) + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \mathbf{v}(t, \mathbf{x}) + H(\mathbf{v}(t, \mathbf{x})). \tag{8}$$

We show that the spectrum of the linear operator in (8) touches the imaginary axis in Section 3, meaning that the weight function needs to be introduced in order to further investigate the stability of the system under perturbations.

For $\mathcal{E}_0$, being the Sobolev spaces $H^k(\mathbb{R}^d)(k = 1, 2, \ldots,$ and we often define $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$), which are suited for the study of nonlinear stability because they are closed under multiplication, we denote the norm in $\mathcal{E}_0$ by $\|\cdot\|_0$. Furthermore, we define the weight function of class $\alpha \in \mathbb{R}$ by

$$\gamma(\mathbf{x}) = \gamma_\alpha(z, x_2, \ldots, x_d) = e^\alpha z, \text{ for } \mathbf{x} = (z, x_2, \ldots, x_d) \in \mathbb{R}^d.$$

For a fixed weight function γ_α, we define $\mathcal{E}_\alpha = \{u : \gamma_\alpha u \in \mathcal{E}_0\}$ with the norm $\|u\|_\alpha = \|\gamma_\alpha u\|_0$. Note that $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$ by this definition. Here and below, we use the fact that $H^k(\mathbb{R}^d)$ can be written as the tensor product $H^k(\mathbb{R}^d) = H^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$. For general results on tensor products and operators on tensor products, refer to [16] (Volume I, Section VIII.10). For ease of notation, we use $H^k_\alpha(\mathbb{R}^d) = \left\{u : e^{\alpha z} u \in H^k(\mathbb{R}^d)\right\}$ to denote the weighted Sobolev space.

Although this weighted functional space solves the problem of the spectral instability of the linear operator, it poses a new difficulty in that the nonlinear terms cannot be controlled in the weighted space. Hence, we introduce a new space using the approach originally proposed in [8] in the context of the Hamiltonian:

$$\mathcal{E} := \mathcal{E}_0 \cap \mathcal{E}_\alpha, \text{ with } \|u\|_{\mathcal{E}} = \max\{\|u\|_0, \|u\|_\alpha\}. \tag{9}$$

We prove the following theorem at the end of Section 3. Specifically, when considered in coordinates moving with fronts, we can show that the steady state of a nonlinear model problem with the form (5) is asymptotically stable in an orbital sense in a carefully chosen exponentially weighted space, i.e., the solution near the steady state converges exponentially to the steady state solution itself in the weighted norm as long as the initial perturbation is sufficiently small in both the weighted and unweighted norm.

Finally, in Section 4, we summarize a number of key features of the system used in the model problem (2), then generalize them into hypotheses; thus, for the general system (1), we can prove the stability of the steady state of a traveling front when it satisfies these hypotheses. Moreover, these hypotheses are often very common in reaction–diffusion systems associated with combustion problems.

Remark 1. *To conclude this section, we explain why the end states* $\mathbf{u}_-$ *and* $\mathbf{u}_+$ *were chosen as* $\mathbf{u}_- = (1/\kappa, 0)$ *and* $\mathbf{u}_+ = (0, 1)$ *for the model system*

$$\begin{cases} u_{1t}(t,\mathbf{x}) = \partial_{zz}u_1(t,\mathbf{x}) + c\partial_z u_1 + u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})),\ u_1, u_2 \in \mathbb{R}, \\ u_{2t}(t,\mathbf{x}) = \epsilon\partial_{zz}u_2(t,\mathbf{x}) + c\partial_z u_2 - \kappa u_2(t,\mathbf{x})g(u_1(t,\mathbf{x})),\ \mathbf{x} \in \mathbb{R}^d, \end{cases} \tag{10}$$

where g is defined in (3)*.*

Let $\Phi = (\phi_1, \phi_2)$ *be a time-independent solution of the model system such that* Φ *satisfies the ODE system*

$$\begin{cases} \partial_{zz}\phi_1(\mathbf{x}) + c\partial_z\phi_1 + \phi_2(\mathbf{x})g(\phi_1(\mathbf{x})) = 0, \\ \epsilon\partial_{zz}\phi_2(\mathbf{x}) + c\partial_z\phi_2 - \kappa\phi_2(\mathbf{x})g(\phi_1(\mathbf{x})) = 0. \end{cases} \tag{11}$$

Here, we are interested in solutions of (11) *that satisfy the boundary conditions at* $z \to \pm\infty$*:*

$$(\phi_1, \phi_2)(-\infty) = (\phi_1^\star, 0),\ (\phi_1, \phi_2)(\infty) = (0, 1).$$

Such solutions represent traveling combustion fronts. Here, the left temperature $\phi_1^\star$ *is an unknown to be determined.*

In the ODE system (11)*, we set* $\phi_3 = \partial_z\phi_1$ *and* $\phi_4 = \partial_z\phi_2$*, and use prime to denote the derivative with respect to z to obtain the following first-order system:*

$$\phi_1' = \phi_3, \tag{12}$$

$$\phi_2' = \phi_4, \tag{13}$$

$$\phi_3' = -(c\phi_3 + \phi_2 g(\phi_1)), \tag{14}$$

$$\phi_4' = -\frac{1}{\epsilon}[c\phi_4 - \kappa\phi_2 g(\phi_1)]. \tag{15}$$

By adding (14) *to* (15) *multiplied by* ϵ/κ*, we obtain the following equation:*

$$\phi_1'' + c\phi_1' + \frac{\epsilon}{\kappa}\phi_2'' + \frac{c}{\kappa}\phi_2' = 0. \tag{16}$$

This expression can be integrated once to produce a function of z that is constant along any traveling wave. We denote this constant by k, meaning that

$$\phi_3 + c\phi_1 + \frac{\epsilon}{\kappa}\phi_4 + \frac{c}{\kappa}\phi_2 = constant := k. \tag{17}$$

For the solution that approaches $(\phi_1, \phi_2, \phi_3, \phi_4) = (0, 1, 0, 0)$ *as* $z \to \infty$*, we must have* $k = \frac{c}{\kappa}$*. Substituting* $k = \frac{c}{\kappa}$ *into Equation* (17)*, we have*

$$\phi_3 = -c\phi_1 - \frac{\epsilon}{\kappa}\phi_4 - \frac{c}{\kappa}\phi_2 + \frac{c}{\kappa} \to 0,\ \phi_1 \to \phi_1^*,\ \phi_2 \to 0 \text{ and } \phi_4 \to 0 \text{ as } z \to -\infty \tag{18}$$

as the steady solution of the system (12)–(15) approaches $(\phi_1, \phi_2, \phi_3, \phi_4) = (\phi_1^\star, 0, 0, 0)$, because $z \to -\infty$. Thus, we necessarily have $\phi_1^\star = \frac{1}{\kappa}$.

3. Stability of the Steady States for the Model Case

In this section, we study the stability of the state $\mathbf{u}_-$ of systems of the model problem (2). This section is organized as follows. We study the spectrum of the operator generated by linearizing (5) with respect to the state in both unweighted and weighted spaces in Section 3.1. The Lipschitz property of the nonlinear term $H(\mathbf{v}(t, \mathbf{x}))$ is shown in Section 3.2, and the stability of the steady-state solution $\mathbf{u}_-$ is proven in Sections 3.3 and 3.4.

3.1. The Setting in the Model Case

Information about the stability of the steady state of system (5) is often disclosed by the information about the spectrum of the linear operator obtained by linearizing (5) with respect to the steady state. Therefore, we first define the linear differential expression in (8) by L:

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_{\mathbf{x}} + c\partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}. \tag{19}$$

We consider a differential operator $\mathcal{L}$ associated with the differential expression L in the Sobolev space $H^k(\mathbb{R}^d)^2$ of vector-valued functions, and throughout we assume that $k \geq [\dfrac{d+1}{2}]$.

Definition 1 ([17])**.** *In work on viscous conservation laws and related equations, a traveling wave is called spectrally stable in $\mathcal{X}$ if the spectrum of $\mathcal{L}$ is contained in $\{\lambda : \operatorname{Re} \lambda < 0\} \cup \{0\}$ and 0 is a simple eigenvalue of $\mathcal{L}$.*

As shown below, the essential spectrum of $\mathcal{L}$ touches the imaginary axis. This prevents $\mathbf{u}_-$ from being stable in the space $H^k(\mathbb{R}^d)^2$, meaning that we have to replace this space by the weighted space $H^k_\alpha(\mathbb{R}^d)^2$, which has an exponential weight with respect to the variable z. In this new space, the nonlinearity loses the local Lipschitz property needed to determine the well-posedness of (8). To regain this, as in [10–12,15], we move on to the intersection space $H^k(\mathbb{R}^d)^2 \cap H^k_\alpha(\mathbb{R}^d)^2$ to perform further analysis.

Remark 2. *We first need information about the spectra of the linear operators associated with (19), which involves several operators in the different spaces considered below . We use the following notation for these operators. If B is a general (2×2) system of n differential expressions, for instance, as in (19), then we use the notation $\mathcal{B} : \mathcal{E}^2_0 \to \mathcal{E}^2_0$ and $\mathcal{B}_\alpha : \mathcal{E}^2_\alpha \to \mathcal{E}^2_\alpha$ to denote the linear operator in $\mathcal{E}^2_0$ and $\mathcal{E}^2_\alpha$, respectively, as provided by the formula $u \to Bu$, with their natural domains. That is, for $k = 0, 1, \cdots$, we use $\mathcal{L} : \mathcal{E}^2_0 \to \mathcal{E}^2_0$ to denote the linear operator provided by the formula $u \mapsto Lu$, for which the domain is $H^{k+2}(\mathbb{R}^d)^2$. We use $\mathcal{L}_\alpha : \mathcal{E}^2_\alpha \to \mathcal{E}^2_\alpha$ to denote the operator in $\mathcal{E}^2_\alpha$ provided by the formula $u \mapsto Lu$, for which the domain is the set of (u_1, u_2), where $\gamma_\alpha u_1, \gamma_\alpha u_2 \in H^{k+2}(\mathbb{R}^d)$. We use the notation $\mathcal{L}_\mathcal{E} : \mathcal{E}^2 \to \mathcal{E}^2$ to denote the linear operator provided by $u \to Lu$, with the domain of $\mathcal{L}_\mathcal{E}$ being the set of (u_1, u_2) satisfying $(u_1, u_2) \in \operatorname{dom}(\mathcal{L}) \cap \operatorname{dom}(\mathcal{L}_\alpha)$, where $\operatorname{dom}(\mathcal{L})$ and $\operatorname{dom}(\mathcal{L}_\alpha)$ are the respective domains defined above.*

First, we use Fourier transform to explore the spectrum of the constant coefficient differential operator $\mathcal{L}$ on $L^2(\mathbb{R}^d)^2$ and the spectrum of the constant coefficient differential operator $\mathcal{L}_\alpha$ on $(L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}))^2$, respectively. We use the following elementary proposition to show that the spectrum of $\mathcal{L}$ on $\mathcal{E}^2_0$ touches the imaginary axis and that the spectrum of $\mathcal{L}_\alpha$ on $\mathcal{E}^2_\alpha$ is away from the imaginary axis.

Proposition 1. *Assume that $\mathcal{L}$ and $\mathcal{L}_\alpha$ are the constant coefficient linear differential operators associated with the differential expression L in (19). On the unweighted space $\mathcal{E}^2_0 = H^k(\mathbb{R}^d)^2$ for all integers $k \geq 0$, we have*

$$\sup\{\operatorname{Re} \lambda : \lambda \in \operatorname{Sp}(\mathcal{L})\} = 0,$$

meaninf that the spectrum of $\mathcal{L}$ touches the imaginary axis. By choosing $\alpha \in (0, c/2)$, we have

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_\alpha)\} < -\nu$$

for some $\nu > 0$, meaning that the spectrum of $\mathcal{L}_\alpha$ is shifted to the left of the imaginary axis on the weighted space $\mathcal{E}_\alpha^2 = (H_\alpha^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}))^2$.

Furthermore, there exists $K > 0$ such that $\|e^{t\mathcal{L}_\alpha}\|_{\mathcal{E}_\alpha^2 \to \mathcal{E}_\alpha^2} \leqslant Ke^{-\nu t}$ for $t \geqslant 0$

Proof. By Lemma 1, as proved below, it is enough to consider the case $k = 0$, that is, to assume that $\mathcal{E}_0 = L^2(\mathbb{R}^d)$. To find $\mathrm{Sp}(\mathcal{L})$ in the unweighted space $\mathcal{E}_0^2$, we can use Fourier transform. From the properties of the Fourier transform (see, e.g., [18], Section 6.5), the operator $\mathcal{L}$ on $L^2(\mathbb{R}^d)^2$ is similar to the operator on $L^2(\mathbb{R}^d)^2$ when multiplying by the matrix-valued function

$$M(\xi) = -(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} + i\xi_1 c\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}, \tag{20}$$

where $\xi = (\xi_1, ..., \xi_d) \in \mathbb{R}^d$. Thus, the spectrum of $\mathcal{L}$ on $L^2(\mathbb{R}^d)^2$ is the closure of the union over $\xi \in \mathbb{R}^d$ of the spectra of the matrices $M(\xi)$. Hence, the spectrum of $\mathcal{L}$ is equal to the closure of the set of $\lambda \in \mathbb{C}$ for which there exists $\xi \in \mathbb{R}^d$ such that

$$\begin{aligned} \det\big(M(\xi) - \lambda I\big) &= \det\Big(-(\xi_1^2 + \xi_2^2 + \cdots + \xi_d^2)\left(\begin{smallmatrix} 1 & 0 \\ 0 & \epsilon \end{smallmatrix}\right) + i\xi_1 cI + \left(\begin{smallmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{smallmatrix}\right)\Big) \\ &= 0. \end{aligned}$$

This is a collection of curves $\lambda = \lambda(\xi)$, where $\lambda(\xi)$ are the eigenvalues of the matrices $M(\xi)$; thus, the spectrum of the operator $\mathcal{L}$ is

$$\begin{aligned} \mathrm{Sp}(\mathcal{L}) &= \bigcup_{\xi \in \mathbb{R}^d} \mathrm{Sp}\begin{pmatrix} -(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 & e^{-\kappa} \\ 0 & -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 - \kappa e^{-\kappa} \end{pmatrix} \\ &= \bigcup_{\xi \in \mathbb{R}^d} (-(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1) \bigcup \bigcup_{\xi \in \mathbb{R}^d} (-\epsilon(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 - \kappa e^{-\kappa}). \end{aligned} \tag{21}$$

This implies that the spectrum of $\mathcal{L}$ in $L^2(\mathbb{R}^d)^2$ touches the imaginary axis when $\xi = (\xi_1, ..., \xi_d) = (0, ..., 0)$.

We now need $\mathrm{Sp}(\mathcal{L}_\alpha)$ on the weighted space $\mathcal{E}_\alpha^2$. First, we define the linear map $N : \mathcal{E}_\alpha \mapsto \mathcal{E}_0$ provided by $Nv = \gamma_\alpha v$; note that, by definition, N is an isomorphism of $\mathcal{E}_\alpha$ on $\mathcal{E}_0$. In particular, we can define a linear operator $\hat{\mathcal{L}} = N\mathcal{L}_\alpha N^{-1}$ on $\mathcal{E}_0^2 = L^2(\mathbb{R}^d)^2$ with the domain $\mathrm{dom}(\hat{\mathcal{L}}) = H^2(\mathbb{R}^d)^2$, as N^{-1} maps $\mathrm{dom}(\hat{\mathcal{L}})$ in $\mathrm{dom}(\mathcal{L}_\alpha)$. As the operator $\hat{\mathcal{L}}$ is similar to $\mathcal{L}_\alpha$ on $\mathcal{E}_\alpha^2$, it has the same spectrum.

In particular, let us consider the operator $\partial_{z,\alpha}$ on $\mathcal{E}_\alpha$ with

$$\mathrm{dom}(\partial_{z,\alpha}) = H_\alpha^1(\mathbb{R}) \otimes H^1(\mathbb{R}^{d-1}).$$

Fix any $v \in H^1(\mathbb{R}^d) = \mathrm{dom}(\hat{\partial_z})$ when $\hat{\partial_z}$ is considered in $L^2(\mathbb{R}^d)$ and $\hat{\partial_z} = N\partial_{z,\alpha}N^{-1}$. Then, temporarily re-denoting $\gamma_\alpha(z) = e^{\alpha z}$, we have

$$\begin{aligned} \partial v = N\partial_{z,\alpha}N^{-1}v = \gamma_\alpha \partial_z(\gamma_{-\alpha}v) &= \gamma_\alpha(\gamma'_{-\alpha}v + \gamma_{-\alpha}\partial_z v) \\ &= \gamma_\alpha(-\alpha\gamma_{-\alpha}v + \gamma_{-\alpha}\partial_z v) \\ &= (\partial_z - \alpha)v. \end{aligned}$$

Denoting $y = (x_2, ..., x_d)$, $\mathbf{x} = (z, y) \in \mathbb{R}^d$; a similar computation shows that for each

$$\mathbf{v} = (v_1, v_2)^T \in \mathrm{dom}\,\hat{\mathcal{L}} = H^2(\mathbb{R}^d)^2 \subset L^2(\mathbb{R}^d)^2,$$

we have

$$\begin{aligned}\hat{\mathcal{L}}\mathbf{v} =& e^{\alpha z}\left(\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}(\Delta_y + \partial_{zz}) + c\partial_z + \begin{pmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{pmatrix}\right)(e^{-\alpha z}\mathbf{v})\\ =& \left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right)\Delta_y\mathbf{v} + \left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right)(\alpha^2\mathbf{v} - 2\alpha\partial_z\mathbf{v} + \partial_{zz}\mathbf{v}) + c(\partial_z\mathbf{v} - \alpha\mathbf{v}) + \left(\begin{smallmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{smallmatrix}\right)\mathbf{v}\\ =& \left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right)\Delta_{\mathbf{x}}\mathbf{v} + \left(cI - 2\alpha\left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right)\right)\partial_z\mathbf{v} + \left(\alpha^2\left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right) - c\alpha I + \left(\begin{smallmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{smallmatrix}\right)\right)\mathbf{v}.\end{aligned}$$

Via Fourier transform, the operator $\hat{\mathcal{L}}$ on $L^2(\mathbb{R}^d)^2$ is similar to the operator of multiplication on $L^2(\mathbb{R}^d)^2$ by the matrix-valued function

$$\begin{aligned}N(\xi) &= -\|\xi\|^2\left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right) + (i\xi_1 c - \alpha c)I + (\alpha^2 - 2i\xi_1\alpha)\left(\begin{smallmatrix}1 & 0\\ 0 & \epsilon\end{smallmatrix}\right) + \left(\begin{smallmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{smallmatrix}\right)\\ &= \left(\begin{smallmatrix}-\|\xi\|^2 + (c-2\alpha)i\xi_1 + \alpha^2 - \alpha c & e^{-\kappa}\\ 0 & -\epsilon\|\xi\|^2 + (c-2\alpha\epsilon)i\xi_1 + \alpha^2\epsilon - c\alpha - \kappa e^{-\kappa}\end{smallmatrix}\right)\end{aligned}$$

where $\|\xi\|^2 = \xi_1^2 + \cdots + \xi_d^2$. Hence,

$$\begin{aligned}\mathrm{Sp}(\mathcal{L}_\alpha) = & \underset{\xi\in\mathbb{R}^d}{\cup}(-(\xi_1^2 + \cdots + \xi_d^2) + (c - 2\alpha)i\xi_1 + \alpha^2 - c\alpha)\\ & \bigcup \underset{\xi\in\mathbb{R}^d}{\cup}(-\epsilon(\xi_1^2 + \cdots + \xi_d^2) + (c - 2\alpha\epsilon)i\xi_1 + \alpha^2\epsilon - c\alpha - \kappa e^{-\kappa}).\end{aligned} \tag{22}$$

Then,

$$\begin{aligned}\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_\alpha)\} &= \sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\hat{\mathcal{L}})\}\\ &= \max(\alpha^2 - c\alpha, \epsilon\alpha^2 - c\alpha - \kappa e^{-\kappa})\\ &= \alpha^2 - c\alpha.\end{aligned}$$

Thus, we conclude that for $\alpha \in (0, c/2)$, we have $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_\alpha)\} < 0$, meaning that the spectrum $\mathrm{Sp}(\mathcal{L}_\alpha)$ on the weighted space $\mathcal{E}_\alpha^2$ is moved to the left of the imaginary axis.

Furthermore, the operator $\mathcal{L}_\alpha$ associated with the differential expression L in (19) generates an analytic semigroup provided that $\epsilon > 0$, and a strongly continuous semigroup provided that $\epsilon = 0$. As shown in [10], in either case $\mathcal{L}$ enjoys the spectral mapping property, that is, the boundary of the spectrum of the semigroup operator $e^{t\mathcal{L}_\alpha}$ is controlled by the boundary of the spectrum of the semigroup generator $\mathcal{L}_\alpha$ for any $\epsilon \geqslant 0$. Then, by the above-mentioned semigroup property (see, e.g., [10] Proposition 4.3), there exists $K > 0$ such that $\|e^{t\mathcal{L}_\alpha}\|_{\mathcal{E}_\alpha^2 \to \mathcal{E}_\alpha^2} \leqslant Ke^{-\nu t}$. □

Lemma 1. *The linear constant coefficient differential operator $\mathcal{L}$ associated with L defined in* (19) *has the same spectrum on $L^2(\mathbb{R}^d)^2$ and on $H^k(\mathbb{R}^d)^2$ for all integers $k > 0$; similarly, the operator $\mathcal{L}_\alpha$ associated with L defined in* (19) *has the same spectrum on $(L_\alpha^2(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}))^2$ and on $(H_\alpha^k(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}))^2$ for all integers $k > 0$.*

Proof. To show that the spectrum of $\mathcal{L}$ in $H^k(\mathbb{R}^d)^2$ is the same as the spectrum of $L^2(\mathbb{R}^d)^2$, we let $\mathcal{F}_1$ denote the Fourier transform acting from $H^k(\mathbb{R}^d)^2$ into $L_m^2(\mathbb{R}^d)^2$, where $L_m^2(\mathbb{R}^d)^2$ is the weighted L^2-space with the standard weight $m(\xi) = (1 + |\xi|_{\mathbb{R}^d}^2)^{k/2}$. By the standard property of the Fourier transform, we have $\mathcal{F}_1\Delta_x = -|\xi|_{\mathbb{R}^d}^2\mathcal{F}_1$ and $\mathcal{F}_1\partial_z = -i\xi_1\mathcal{F}_1$. Thus, $\mathcal{F}_1\mathcal{L} = M\mathcal{F}_1$ for a matrix-valued function $M = M(\xi)$ obtained from (19) by replacing Δ_x by $-|\xi|_{\mathbb{R}^d}^2$ and ∂_z by $-i\xi_1$.

On the other hand, the operator of multiplication by $m(\cdot)$ is an isomorphism of $L_m^2(\mathbb{R}^d)^2$ onto $L^2(\mathbb{R}^d)^2$. Let us denote by $\mathcal{L}_{H^k}$ the operator $\mathcal{L}$ associated with L on the space $H^k(\mathbb{R}^d)^2$ and by $\mathcal{L}_{L^2}$ the operator $\mathcal{L}$ associated with L on the space $L^2(\mathbb{R}^d)^2$. Per the previous paragraph, we then have $m\mathcal{F}_1\mathcal{L}_{H^k} = Mm\mathcal{F}_1$. Here and below we use a slight abbreviation of this notation; properly written, $u \in \mathrm{dom}(\mathcal{L}_{H^k})$ implies $m\mathcal{F}_1 u \in \mathrm{dom}(M)$ and $m\mathcal{F}_1\mathcal{L}_{H^k}u = Mm\mathcal{F}_1 u$ for all $u \in \mathrm{dom}(\mathcal{L}_{H^k})$.

We remark that the operator of multiplication by $-i\xi_j, j = 1, ..., d$ on $L^2(\mathbb{R}^d)$ is similar to the operator of differentiation ∂_{x_j} on $L^2(\mathbb{R}^d)$ via Fourier transform $\mathcal{F}_2$. This implies that $\mathcal{F}_2\mathcal{L}_{L^2} = M\mathcal{F}_2$ with the same matrix-valued function M as above. It follows that

$$\mathcal{L}_{H^k} = (m\mathcal{F}_1)^{-1}Mm\mathcal{F}_1 = (m\mathcal{F}_1)^{-1}(\mathcal{F}_2\mathcal{L}_{L^2}\mathcal{F}_2^{-1})(m\mathcal{F}_1); \tag{23}$$

therefore, the spectrum of $\mathcal{L}$ on $H^k(\mathbb{R}^d)^2$ is the same as the spectrum of $\mathcal{L}$ on $L^2(\mathbb{R}^d)^2$, as the operators on $H^k(\mathbb{R}^d)^2$ and $L^2(\mathbb{R}^d)^2$ are similar.

By analogous argument, the spectrum of $\mathcal{L}_\alpha$ on $(L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}))^2$ is the same as the spectrum of $\mathcal{L}_\alpha$ on $(H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}))^2$. $\square$

Remark 3. *Recall that we denote $y = (x_2, ..., x_d)$. Let Δ_y be the operator provided by the differential expression $\partial^2_{x_2} + \cdots + \partial^2_{x_d}$, where the domain of Δ_y on $H^k(\mathbb{R}^{d-1})$ is the set of u such that $u \in H^{k+2}(\mathbb{R}^{d-1})$. We denote by $\mathcal{L}_1 : H^k(\mathbb{R})^2 \to H^k(\mathbb{R})^2$ and $\mathcal{L}_{1,\alpha} : H^k_\alpha(\mathbb{R})^2 \to H^k_\alpha(\mathbb{R})^2$ respectively the operators provided by the differential expression $\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}\partial_{zz} + c\partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}$ acting on unweighted and weighted spaces. The operator $\mathcal{L}_\alpha$ on $\mathcal{E}^2_\alpha = (H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}))^2$ can be written as $\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y$. We have yet another approach to prove Propositon 1 using [16] (Volume IV, Theorem XIII.34, Theorem XIII.35, and Corollary 1). Indeed, because $\mathcal{L}_{1,\alpha}$ and Δ_y are the generators of bounded semigroups on $H^k_\alpha(\mathbb{R})^2$ and $H^k(\mathbb{R}^{d-1})$, respectively, we have the following (see [16], Volume IV):*

$$\mathrm{Sp}(\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y) = \mathrm{Sp}(\mathcal{L}_{1,\alpha}) + \mathrm{Sp}(\Delta_y). \tag{24}$$

Thus, $\mathrm{Sp}(\mathcal{L}_\alpha) = \mathrm{Sp}(\mathcal{L}_{1,\alpha}) + \mathrm{Sp}(\Delta_y)$. It is easy to see that the spectrum of Δ_y on $H^k(\mathbb{R}^{d-1})$ is the non-negative semiline $(-\infty, 0]$, and the spectrum of $\mathcal{L}_{1,\alpha}$ on $H^k_\alpha(\mathbb{R})^2$ satisfies $\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_{1,\alpha})\} < -\nu$ for some $\nu > 0$; thus, Proposition 1 is proven. Moreover, the same argument shows that if Γ is the curve that bounds the spectrum of $\mathcal{L}_{1,\alpha}$ on the right, then $\mathrm{Sp}(\mathcal{L}_\alpha)$ is the entire solid part of the plane bounded by Γ. We use Figure 1 to illustrate the spectrum of the linear operator $\mathcal{L}$ on different spaces; in the weighted space, the spectrum is away from the imaginary axis and the operator Δ_y extends in a semiline to negative infinity at each point on the original spectrum.

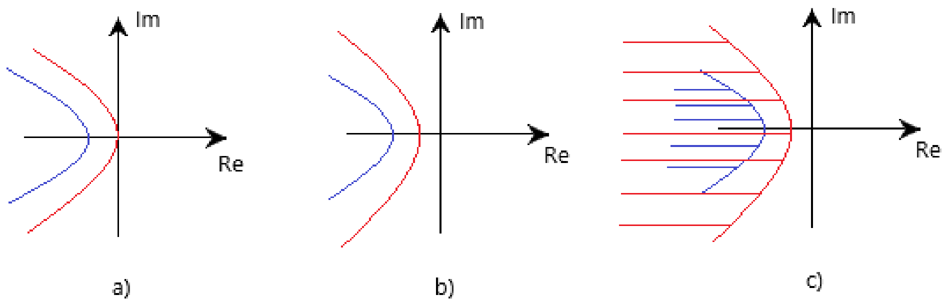

Figure 1. (**a**) spectrum of the operator $\mathcal{L}_1$ on $H^k(\mathbb{R})^2$, the red curve corresponds to the operator $\partial_{zz} + c\partial_z$, while the blue curve corresponds to the operator $\epsilon\partial_{zz} + c\partial_z - \kappa e^{-\kappa}$; (**b**) spectrum of the operator $\mathcal{L}_{1,\alpha}$ on $H^k_\alpha(\mathbb{R})^2$; (**c**) spectrum of the operator $\mathcal{L}_\alpha$ on $H^k_\alpha(\mathbb{R}^d)^2$.

Now, notice that the differential expression L in (19) has the following triangular structure:

$$L = \begin{pmatrix} \Delta_{\mathbf{x}} + c\partial_z & e^{-\kappa} \\ 0 & \epsilon\Delta_{\mathbf{x}} + c\partial_z - \kappa e^{-\kappa} \end{pmatrix}. \tag{25}$$

Let

$$L^{(1)} = \Delta_{\mathbf{x}} + c\partial_z; \tag{26}$$
$$L^{(2)} = \epsilon\Delta_{\mathbf{x}} + c\partial_z - \kappa e^{-\kappa}, \tag{27}$$

and for $i = 1,2$ let $\mathcal{L}^{(i)}$ be the operator on $H^k(\mathbb{R}^d)$ defined by $v_i \mapsto L^{(i)}v_i$ (with the domain of $\mathcal{L}^{(i)}$ being $H^{k+2}(\mathbb{R}^d)$) for $k = 0,1,2,\dots$.

Lemma 2. *Consider the operators $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ on $H^k(\mathbb{R}^d)$ defined by the differential expressions $L^{(1)}$ and $L^{(2)}$ provided in (26) and (27).*

(1) The operator $\mathcal{L}^{(1)}$ generates a bounded strongly continuous semigroup on $H^k(\mathbb{R}^d)$;

(2) The operator $\mathcal{L}^{(2)}$ satisfies $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}^{(2)})\} < 0$ on $H^k(\mathbb{R}^d)$;

(3) The following is true on $H^k(\mathbb{R}^d)$:

- *(a) $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}^{(1)})\} \le 0$;*
- *(b) There exist $K > 0$ and $\rho > 0$ such that for the strongly continuous semigroup $\{e^{t\mathcal{L}^{(2)}}\}_{t\ge 0}$ we have $\|e^{t\mathcal{L}^{(2)}}\|_{H^k(\mathbb{R}^d)\to H^k(\mathbb{R}^d)} \le Ke^{-\rho t}$ for all $t \ge 0$.*

Proof. As in Lemma 1, we can prove that the operators $\mathcal{L}^{(i)}$, $i = 1,2$ have the same spectrum on $H^k(\mathbb{R}^d)$ and on $L^2(\mathbb{R}^d)$.

Using the Fourier transform, we find that the spectrum of $\mathcal{L}^{(1)}$ on $L^2(\mathbb{R}^d)$ is the union of the curves $\lambda_1(\xi) = -(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1$ for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Thus, $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}^{(1)})\} \le 0$ on $L^2(\mathbb{R})$, which proves (3)(a). Per the proof of Proposition A.1(1) in [11], the operator $\mathcal{L}^{(1)}$ generates a bounded semigroup on $L^2(\mathbb{R}^d)$. The operators on $H^k(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d)$ associated with the same constant–coefficient differential expression are similar (see (23)); therefore, the semigroup they generate are similar, and (1) is proved.

The spectrum of $\mathcal{L}^{(2)}$ on $L^2(\mathbb{R}^d)$ is the union of the curves $\lambda_2(\xi) = -\epsilon(\xi_1^2 + \cdots + \xi_d^2) + ci\xi_1 - \kappa e^{-\kappa}$ for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$; therefore, $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}^{(2)})\} < 0$ on $L^2(\mathbb{R}^d)$, and per Lemma 1 on $H^k(\mathbb{R}^d)$ as well, proving (2).

The assertion of (3)(b) is a direct consequence of (2); see [11] (Lemma 3.13). □

3.2. Nonlinear Terms in the Model Case

In this subsection, we study the nonlinear terms defined in (7) and prove that the nonlinearity is locally Lipschitz on the intersection space $\mathcal{E}$.

Recall that we introduced the nonlinear term of system (8) in Formula (7), that is,

$$\begin{aligned} H(\mathbf{v}(t,\mathbf{x})) &= f(\mathbf{u}_- + \mathbf{v}(t,\mathbf{x})) - f(\mathbf{u}_-) - \partial_{\mathbf{u}} f(\mathbf{u}_-)v(t,\mathbf{x}) \\ &= f\left(\begin{pmatrix} 1/\kappa + v_1 \\ v_2 \end{pmatrix}\right) - \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= \begin{pmatrix} v_2(e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa}) \\ -\kappa v_2(e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa}) \end{pmatrix}. \end{aligned} \tag{28}$$

To obtain the Lipschitz property of the nonliner term on the multidimensional spaces $H^k(\mathbb{R}^d)$ and $H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$, we need to use space $\mathcal{E}$ as defined in Equation (9).

It is convenient to write $H(\mathbf{v})$ as follows:

$$H(\mathbf{v}) = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix}(g(\frac{1}{\kappa} + v_1) - g(\frac{1}{\kappa}))v_2, \tag{29}$$

where $\mathbf{v} = (v_1, v_2)$ and $g(\cdot)$ is defined as in (2).

The proofs below are based on the fact that Sobolev embedding yields the inequality

$$||uv||_{H^k(\mathbb{R}^d)} \le C||u||_{H^k(\mathbb{R}^d)}||v||_{H^k(\mathbb{R}^d)} \tag{30}$$

for $2k > d$; see [19] (Theorem 4.39). We begin with a few elementary facts.

Lemma 3. *Assume that $k \geq [\frac{d+1}{2}]$ and consider $\mathcal{E}_0 = H^k(\mathbb{R}^d)$. Then, the following assertions hold.*

(1) If $u, v \in \mathcal{E}_0$, then $uv \in \mathcal{E}_0$, and there exists a constant $C > 0$ such that $\|uv\|_0 \leq C\|u\|_0\|v\|_0$.
(2) If $u, v \in \mathcal{E}$, then $uv \in \mathcal{E}_\alpha$, and there exists a constant $C > 0$ such that $\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha$.
(3) If $u, v \in \mathcal{E}$, then $uv \in \mathcal{E}$, and there exists a constant $C > 0$ such that $\|uv\|_{\mathcal{E}} \leq C\|u\|_{\mathcal{E}}\|v\|_{\mathcal{E}}$.

Proof. Assertion (1) is in fact the Sobolev embedding inequality (30). Assertion (2) can be proven using (30), as

$$\|uv\|_\alpha = \|\gamma_\alpha uv\|_0 \leq C\|u\|_0\|\gamma_\alpha v\|_0 = C\|u\|_0\|v\|_\alpha.$$

To show (3), let $u, v \in \mathcal{E}$. Then, per (1),

$$\|uv\|_0 \leq C\|u\|_0\|v\|_0 \leq C\|u\|_{\mathcal{E}}\|v\|_{\mathcal{E}},$$

and per (2),

$$\|uv\|_\alpha \leq C\|u\|_0\|v\|_\alpha \leq C\|u\|_{\mathcal{E}}\|v\|_{\mathcal{E}}.$$

Therefore, $uv \in \mathcal{E}$ and $\|uv\|_{\mathcal{E}} \leq C\|u\|_{\mathcal{E}}\|v\|_{\mathcal{E}}$. □

The nonlinearities of type (29) are a combination of the Nemytskij-type operator $v_1 \mapsto g(1/\kappa + v_1)$ and multiplication operator by v_2. In what follows, we need to establish both their local Lipschitz properties and more general operators of the type $v \mapsto m(v(\cdot))v(\cdot)$, where $m(\cdot)$ is a given function and $v \in H^k(\mathbb{R}^d)$. One-dimensional results of this kind can be found in [11] (Proposition 7.2). We present an analogue of the proof of [11] (Proposition 7.2)t in [15] (Appendix A); see Lemma 4 below.

Lemma 4. *Assume $k \geq [\frac{d+1}{2}]$ and let $m : (q, u) \mapsto m(q, u) \in \mathbb{R}$ be a function from $C^{k+1}(\mathbb{R}^2)$. Consider the formula*

$$(q(\mathbf{x}), u(\mathbf{x}), v(\mathbf{x})) \mapsto m(q(\mathbf{x}), u(\mathbf{x}))v(\mathbf{x}), \tag{31}$$

where $q(\cdot), u(\cdot), v(\cdot) : \mathbb{R}^d \mapsto \mathbb{R}$, and the variable $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d$.

(1) Formula (31) defines a mapping from $H^k(\mathbb{R}^d) \times \mathcal{E}_0^2$ to $\mathcal{E}_0$ that is locally Lipschitz on any set with the form $\{(q, u, v) : \|q\|_0 + \|u\|_0 + \|v\|_0 \leq K\}$.
(2) Formula (31) defines a mapping from $H^k(\mathbb{R}^d) \times \mathcal{E}^2$ to $\mathcal{E}$ that is locally Lipschitz on any set with the form $\{(q, u, v) : \|q\|_0 + \|u\|_{\mathcal{E}} + \|v\|_{\mathcal{E}} \leq K\}$.

By dropping q from Lemma 4, we record the following corollary that can be used to study the components of the map $H(\cdot)$ from (29).

Corollary 1. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, and let $\mathcal{E}_\alpha$ and $\mathcal{E}$ be defined accordingly. If $k \geq [\frac{d+1}{2}]$ and $m(\cdot) \in C^\infty(\mathbb{R})$, then the formula*

$$v(\mathbf{x}) \mapsto m(v(\mathbf{x}))v(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

defines mappings from $\mathcal{E}_0$ to $\mathcal{E}_0$ and from $\mathcal{E}$ to $\mathcal{E}$. The first is locally Lipschitz on any set with the form $\{v : \|v\|_0 \leq K\}$, while the second is locally Lipschitz on any set with the form $\{v : \|v\|_{\mathcal{E}} \leq K\}$.

Proposition 2. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, and let $\mathcal{E}_\alpha$ and $\mathcal{E}$ be defined accordingly. Let $k \geq [\frac{d+1}{2}]$, and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and consider the formula*

$$H(\mathbf{v}) = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)$$

as provided in (28).

(1) *$H(\cdot)$ defines a mapping from $\mathcal{E}_0^2$ to $\mathcal{E}_0^2$ that is locally Lipschitz on any set with the form $\{v : ||v||_0 \leq K\}$.*
(2) *$H(\cdot)$ defines a mapping from $\mathcal{E}^2$ to $\mathcal{E}^2$ that is locally Lipschitz on any set with the form $\{v : ||v||_{\mathcal{E}} \leq K\}$.*

Proof. It can be shown that $g(\cdot) \in C^\infty(\mathbb{R})$ is a smooth bounded function. Let

$$m(v_1) = g(v_1 + 1/\kappa) - g(1/\kappa);$$

then, $m(\cdot)$ is a smooth and bounded function. By applying Corollary 1 to the components of the vector-valued map H, we finish the proof. □

Proposition 3. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$, and let $\mathcal{E}_\alpha$ and $\mathcal{E}$ be defined accordingly. Let $k \geq [\frac{d+1}{2}]$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, and consider the formula*

$$H(\mathbf{v}) = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)$$

provided in (28).

(1) *If $\mathbf{v} \in \mathcal{E}^2$, then there exists a constant $C_K > 0$ such that*

$$||H(\mathbf{v})||_\alpha \leq C_K||\mathbf{v}||_0||\mathbf{v}||_\alpha$$

on any set with the form $\{v : ||\mathbf{v}||_{\mathcal{E}} \leq K\}$.

(2) *If $\mathbf{v} \in \mathcal{E}^2$, then there exists a constant $C_K > 0$ such that*

$$||H(\mathbf{v})||_{\mathcal{E}} \leq C_K||\mathbf{v}||_{\mathcal{E}}^2$$

on any set with the form $\{v : ||\mathbf{v}||_{\mathcal{E}} \leq K\}$.

Proof. Note that

$$g(v_1 + 1/\kappa) - g(1/\kappa) = \int_0^t g'(1/\kappa + tv_1)dt\, v_1.$$

Because $\int_0^t g'(1/\kappa + tv_1)dt$ is a smooth function, from Corollary 1 we have

$$\|g(v_1 + 1/\kappa) - g(1/\kappa)\|_0 \leq C_K\|v_1\|_0.$$

Note that $\|v_1\|_0 \leq \|\mathbf{v}\|_0$ and $\|v_2\|_\alpha \leq \|\mathbf{v}\|_\alpha$, because v_1 and v_2 are components of the vector $\mathbf{v}$. Then, (1) holds, because

$$\begin{aligned}\|H(\mathbf{v})\|_\alpha = \|\gamma_\alpha H(\mathbf{v})\|_0 &= \left\|\gamma_\alpha \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)\right\|_0 \\ &\leq C\|\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)\|_0\|\gamma_\alpha v_2\|_0 \\ &\leq C_K\|v_1\|_0\|v_2\|_\alpha \leq C_K||\mathbf{v}||_0||\mathbf{v}||_\alpha.\end{aligned}$$

Similarly, using the fact that $\|v_2\|_0 \leq \|\mathbf{v}\|_0$, we have

$$\begin{aligned}\|H(\mathbf{v})\|_0 &= \left\|\begin{pmatrix} 1 \\ -\kappa \end{pmatrix} v_2\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)\right\|_0 \\ &\leq C\|\big(g(v_1 + 1/\kappa) - g(1/\kappa)\big)\|_0\|v_2\|_0 \\ &\leq C_K\|v_1\|_0\|v_2\|_0 \leq C_K||\mathbf{v}||_0||\mathbf{v}||_0,\end{aligned}$$

thus,

$$
\begin{aligned}
||H(\mathbf{v})||_{\mathcal{E}} &= \max\{||H(\mathbf{v})||_0, ||H(\mathbf{v})||_\alpha\} \\
&\le \max\{C_K||\mathbf{v}||_0||\mathbf{v}||_0, C_K||\mathbf{v}||_0||\mathbf{v}||_\alpha\} \\
&\le C_K||\mathbf{v}||_{\mathcal{E}}||\mathbf{v}||_{\mathcal{E}},
\end{aligned}
$$

and (2) is proved. □

3.3. Stability of the Steady State of the Planar Front in the Model Case

In this subsection, we prove the stability of the end state $\mathbf{u}_- = (1/\kappa, 0)$ of (5). From Propositions 1 and 2, we know that with initial data $\mathbf{v}^0 \in \mathcal{E}^2$ there is a unique mild solution $\mathbf{v}(t, \mathbf{v}^0)$ to system (8) defined for $t \in [0, t_{max}(v))$, where $0 < t_{max}(v)) \le \infty$ (see, e.g., [20], Theorem 6.1.4). The set $\{(t, \mathbf{v}^0) \in \mathbb{R}_+ \times \mathcal{E}^2 : 0 \le t < t_{max}(\mathbf{v}))\}$ is open in $\mathbb{R}_+ \times \mathcal{E}^2$, and the map $(t, \mathbf{v}^0) \mapsto \mathbf{v}(t, \mathbf{v}^0)$ from this set to $\mathcal{E}^2$ is continuous (see, e.g., [21], Theorem 46.4). We summarize these facts as follows.

Proposition 4. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ with $k \ge [\frac{d+1}{2}]$. Then, for each $\delta > 0$, if $0 < \gamma < \delta$, there exists $T(\gamma, \delta)$ depending on γ and δ with $0 < T(\gamma, \delta) \le \infty$, such that the following is true. If $\mathbf{v}^0 \in \mathcal{E}^2$ satisfies*

$$||\mathbf{v}^0||_{\mathcal{E}} \le \gamma \tag{32}$$

and $0 \le t < T$, the solution $\mathbf{v}(t) \in \mathcal{E}^2$ of (8) is defined and satisfies

$$||\mathbf{v}(t)||_{\mathcal{E}} \le \delta. \tag{33}$$

We can then prove the following proposition to show that $\mathbf{v}(t, \mathbf{v}^0) \in \mathcal{E}^2$ is exponentially decaying in the weighted norm when $\mathbf{v}^0$ is small in $\mathcal{E}^2$. We first establish the exponential decay of the solutions of (8) on $H^k_\alpha(\mathbb{R}^d)^2$.

Proposition 5. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ with $k \ge [\frac{d+1}{2}]$, choosing $\nu > 0$, as in Proposition 1. Then, there exist $\delta_1 > 0$ and $K_1 > 0$ such that the following is true for every $\delta \in (0, \delta_1)$ and every γ with $0 < \gamma < \delta$. Let $\mathbf{v}^0 \in \mathcal{E}^2$, satisfying (32), such that $\mathbf{v}(t)$ satisfies (33) for $0 \le t < T(\delta, \gamma)$. Then,*

$$||\mathbf{v}(t)||_\alpha \le K_1 e^{-\nu t}||\mathbf{v}^0||_\alpha \quad \text{for } 0 \le t < T(\delta, \gamma). \tag{34}$$

Proof. Because $\mathbf{v}(t)$ is a mild solution of (8) on $\mathcal{E}^2$, this satisfies the integral equation

$$\mathbf{v}(t) = e^{t\mathcal{L}_\varepsilon}\mathbf{v}^0 + \int_0^t e^{(t-s)\mathcal{L}_{\mathcal{E}}} N(\mathbf{v}(s))\mathbf{v}(s) ds. \tag{35}$$

Because by assumption $\mathbf{v}^0 \in \mathcal{E}^2$, by Proposition 3 it is clear that $N(\mathbf{v})\mathbf{v}$ is in $H^k_\alpha(\mathbb{R}^d)^2$, thus, we have

$$e^{t\mathcal{L}_{\mathcal{E}}}\mathbf{v}^0 = e^{t\mathcal{L}_\alpha}\mathbf{v}^0 \text{ and } e^{(t-s)\mathcal{L}_{\mathcal{E}}} N(\mathbf{v}(s))\mathbf{v}(s) = e^{(t-s)\mathcal{L}_\alpha} N(\mathbf{v}(s))\mathbf{v}(s)$$

Next, we replace $\mathcal{L}_{\mathcal{E}}$ by $\mathcal{L}_\alpha$ in (35) and choose $\bar{\nu} > \nu > 0$ such that

$$\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\alpha)\} < -\bar{\nu} := -k\nu,$$

for some $k = \bar{\nu}/\nu > 1$ and close to 1. Then, per Proposition 1 there exists $K_1 > 0$ such that $\left\|e^{t\mathcal{L}_\alpha}\right\|_{\mathcal{E}^2 \to \mathcal{E}^2} \leqslant K_1 e^{-\bar{\nu}t}$ for all $t \geqslant 0$.

Choosing any $\delta' > 0$, for any γ such that $0 < \gamma < \delta'$, if $\left\|\mathbf{v}^0\right\|_{\mathcal{E}} < \gamma$, per Proposition 4 we can say that $\|\mathbf{v}(s)\|_{\mathcal{E}} < \delta'$ for all $s \in (0, T(\delta', \gamma))$.

With the aid of Propostion 3 (1), there exists a constant $C_{\delta'} > 0$ depending on δ' such that, for $\|\mathbf{v}(s)\|_{\mathcal{E}} \leqslant \delta'$, when $s \in (0, T(\delta', \gamma))$ it follows that

$$\|\mathbf{v}(t)\|_\alpha \leqslant K_1 e^{-\bar{\nu}t}\left\|\mathbf{v}^0\right\|_\alpha + \int_0^t K_1 e^{-\bar{\nu}(t-s)} C_{\delta'}\|\mathbf{v}(s)\|_0\|\mathbf{v}(s)\|_\alpha ds$$

For each $\delta < \delta'$ and $0 < \gamma < \delta$, if $\|\mathbf{v}^0\|_{\mathcal{E}} < \gamma$, then $\|\mathbf{v}(s)\|_{\mathcal{E}} < \delta$ for all $s \in (0, T(\delta,\gamma))$. Again, per Proposition 4,

$$\|\mathbf{v}(t)\|_\alpha \leqslant K_1 e^{-\bar{\nu}t}\left\|\mathbf{v}^0\right\|_\alpha + K_1 C_{\delta'}\delta \int_0^t e^{-\bar{\nu}(t-s)}\|\mathbf{v}(s)\|_\alpha ds$$

Applying Gronwall's inequality for the function $e^{\bar{\nu}t}||\mathbf{v}(t)\|_\alpha$, we can find that the inequality

$$e^{\bar{\nu}t}\|\mathbf{v}(t)\|_\alpha \leqslant K_1\left\|\mathbf{v}^0\right\|_\alpha + K_1 C_{\delta'}\delta \int_0^t e^{\bar{\nu}s}\|\mathbf{v}(s)\|_\alpha ds$$

implies, by Gronwall's inequality, that

$$\|\mathbf{v}(t)\|_\alpha \leqslant K_1\left\|\mathbf{v}^0\right\|_\alpha e^{K_1 C_{\delta'}\delta t - \bar{\nu}t}.$$

By choosing $\delta_1 < \min\left\{\delta', (k-1)\frac{\nu}{K_1 C_{\delta'}}\right\}$, we can conclude that (34) holds for any $\delta \in (0, \delta_1)$. $\square$

On the unweighted space $\mathcal{E}_0^2 = H^k(\mathbb{R}^d)^2$, we can rewrite system (8) as follows:

$$v_{1t} = L^{(1)}v_1 + e^{-\kappa}v_2 + H_0(\mathbf{v}), \tag{36}$$
$$v_{2t} = L^{(2)}v_2 - \kappa H_0(\mathbf{v}), \tag{37}$$

where $H_0(\mathbf{v}) = v_2\big(g(v_1 + 1/\kappa) - g(v_1)\big)$. Using Propositions 2 and 3, we conclude that $H_0(\cdot)$ defines a mapping from $\mathcal{E}_0^2$ to $\mathcal{E}_0$ that is locally Lipschitz on any set with the form $\{\mathbf{v} : \|\mathbf{v}\|_0 \leq K\}$ and $||H_0(\mathbf{v})||_0 \leq C_K||\mathbf{v}||_0^2$. Therefore, we obtain the following estimate.

Proposition 6. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ with $k \geq [\frac{d+1}{2}]$. Choose $\rho > 0$ as in Lemma 2 (3)(b), and δ_1 as per Proposition 5. Assuming that $\nu < \rho$, where ν are chosen as in Proposition 1, there exist $\delta_2 \in (0, \delta_1)$ and $C_1 > 0$ such that for every $\delta \in (0, \delta_2)$ and every γ with $0 < \gamma < \delta$, then it is true that if $0 \leq t < T(\delta, \gamma)$ and $\mathbf{v}^0 \in \mathcal{E}^2$ satisfies (32) such that the solution $\mathbf{v}(t) \in \mathcal{E}^2$ of (8) satisfies (33), the following estimates hold:*

$$||v_1(t)||_0 \leq C_1||\mathbf{v}^0||_{\mathcal{E}}, \tag{38}$$
$$||v_2(t)||_0 \leq C_1 e^{-\rho t}||\mathbf{v}^0||_{\mathcal{E}}. \tag{39}$$

Proof. We note that (v_1, v_2) is the solution of (36) and (37) with initial values (v_1^0, v_2^0) at $t = 0$, that is $(v_1, v_2)(t) = (v_1, v_2)\big(t, v_1^0, v_2^0\big)$. With the help of Proposition 3 (2), we can find a constant $C_{\delta_1} > 0$, meaning that

$$\|H_1(v_1, v_2)\|_0 \leqslant C_{\delta_1}\|v_1\|_0\|v_2\|_0, \tag{40}$$

and

$$\|H_2(v_1, v_2)\|_0 = \|-\kappa H_1(\mathbf{v})\|_0 \leqslant C_{\delta_1}\|v_1\|_0\|v_2\|_0$$

when $\|\mathbf{v}\|_0 \leqslant \delta_1$. The solution of (37) in $H^k(\mathbb{R}^d)$ can be written as

$$v_2(t) = e^{t\mathcal{L}_2}v_2^0 + \int_0^t e^{(t-s)\mathcal{L}_2}H_2(v_1(s), v_2(s))ds.$$

Then, we choose some $\bar{\rho} > \rho > 0$ and $k = \bar{\rho}/\rho > 1$ such that

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \mathrm{Sp}(\mathcal{L}_2)\} < -\bar{\rho} := -k\rho.$$

By Lemma 2 (3), there exists $K_2 > 0$ such that $\left\|e^{t\mathcal{L}_2}\right\|_{H^k(\mathbb{R}^d)\to H^k(\mathbb{R}^d)} \leqslant K_2 e^{-\bar{\rho}t}$. For each $\delta \in (0, \delta_1)$ and $\gamma \in (0, \delta)$, if $\left\|\mathbf{v}^0\right\|_{\mathcal{E}} \leqslant \gamma$,

$$\|v_1(s)\|_0 \leqslant \|v_1(s)\|_{\mathcal{E}} \leqslant \|\mathbf{v}(s)\|_{\mathcal{E}} \leqslant \delta.$$

Following Proposition 4, we obtain the following estimate for $v_2(t)$ using (37):

$$\begin{aligned}\|v_2(t)\|_0 &\leqslant K_2 e^{-\bar{\rho}t}\left\|v_2^0\right\|_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)} C_{\delta_1}\|v_1(s)\|_0\|v_2(s)\|_0 d\, s \\ &\leqslant K_2 e^{-\bar{\rho}t}\left\|v_2^0\right\|_0 + \int_0^t K_2 e^{-\bar{\rho}(t-s)} C_{\delta_1}\delta\|v_2(s)\|_0 ds.\end{aligned}$$

We then calculate

$$\begin{aligned}e^{\bar{\rho}t}\|v_2(t)\|_0 &\leqslant K_2\left\|v_2^0\right\|_{\mathcal{E}} + K_2 C_{\delta_1}\delta\int_0^t e^{\bar{\rho}s}\|v_2(s)\|_0 d\, s \\ &\leqslant K_2\left\|\mathbf{v}^0\right\|_{\mathcal{E}} + K_2 C_{\delta_1}\delta\int_0^t e^{\bar{\rho}s}\|v_2(s)\|_0 ds.\end{aligned}$$

By applying Gronwall's inequality to $e^{\bar{\rho}t}\|v_2(t)\|_0$, we infer that

$$\|v_2(t)\|_0 \leqslant K_2\left\|\mathbf{v}^0\right\|_{\mathcal{E}} e^{K_2 C_{\delta_1}\delta t - \bar{\rho}t}.$$

Let $\delta_2 < \min\left(\delta_1, \frac{(k-1)\rho}{K_2 C_{\delta_1}}\right)$; then, for $\delta < \delta_2$ it follows that

$$\|v_2(t)\|_0 \leqslant K_2\left\|\mathbf{v}^0\right\|_{\mathcal{E}} e^{-\rho t} \text{ for all } t \in [0, T(\delta,\gamma)),$$

which proves (39). Next, we proceed to prove (38). The solution of (36) in $H^k(\mathbb{R}^d)$ satisfies

$$v_1(t) = e^{t\mathcal{L}_1}v_1^0 + \int_0^t e^{(t-s)\mathcal{L}_1}\left(e^{-\kappa}v_2(s) + H_1(v_1(s), v_2(s)\right)ds.$$

First, because $\mathcal{L}_1$ generates a bounded semigroup, per Lemma 2 (1) there exists a constant $K_3 > 0$ such that $\|e^{t\mathcal{L}_1}\|_{H^k(\mathbb{R}^d)\to H^k(\mathbb{R}^d)} \leqslant K_3$. Using (40) and the fact that

$$\left\|e^{-\kappa}v_2(s)\right\|_0 \leqslant \|v_2(s)\|_0$$

for $\kappa > 0$, we infer that

$$\|v_1(t)\|_0 \leqslant K_3\left\|v_1^0\right\|_0 + \int_0^t \left(K_3 C_{\delta_1}\|v_2(s)\|_0\|v_1(s)\|_0 + K_3\|v_2(s)\|_0\right)ds.$$

Furthermore, using the fact that $\|v_1(s)\|_0 \leqslant \|\mathbf{v}(s)\|_0 \leqslant \|\mathbf{v}(s)\|_{\mathcal{E}} < \delta < \delta_2$, for a constant $C_{\delta_1,\delta_2} > 0$ independent of δ we have

$$\begin{aligned}\|v_1(t)\|_0 &\leqslant K_3\left\|v_1^0\right\|_{\mathcal{E}} + \int_0^t K_3\left(C_{\delta_1}\|v_1(s)\|_0 + 1\right)\|v_2(s)\|_0 d\, s \\ &\leqslant K_3\left\|v_1^0\right\|_{\mathcal{E}} + \int_0^t K_3 C_{\delta_1,\delta_2}\|v_2(s)\|_0 ds.\end{aligned}$$

Then, we use (39) to obtain

$$\begin{aligned}\|v_1(t)\|_0 &\leqslant K_3\left\|\mathbf{v}^0\right\|_{\mathcal{E}} + \int_0^t K_2 K_3 C_{\delta_1,\delta_2} e^{-\rho s}\left\|\mathbf{v}^0\right\|_{\mathcal{E}} d\, s \\ &\leqslant K_3\left\|\mathbf{v}^0\right\|_{\mathcal{E}} + K_2 K_3 C_{\delta_1,\delta_2}\left\|\mathbf{v}^0\right\|_{\mathcal{E}}\int_0^t e^{-\rho s} d\, s \\ &\leqslant C_2\left\|\mathbf{v}^0\right\|_{\mathcal{E}}\end{aligned}$$

for some $C_2 > 0$. In conclusion, there exists a constant $C_1 > 0$ such that the inequalities (38) and (39) hold for $\delta \in (0, \delta_2)$ and $\gamma \in (0, \delta)$ when $t \in [0, T(\delta,\gamma))$. $\square$

3.4. Proof of Theorem 1

In this subsection, we present the main proof of the stability of the end state $\mathbf{u}_-$ of (5) in $\|\cdot\|_{\mathcal{E}}$. The proof relies on the following bootstrap argument based on Propositions 4–6. These propositions yield the existence of constants $\delta_0 > 0$ and $C_{\delta_0} > 0$, meaning that for every $\delta \in (0, \delta_0)$ and every $\gamma \in (0, \delta)$ there exists $T(\delta, \gamma)$ such that for every $t \in [0, T(\delta, \gamma))$ the inequalities

$$\|\mathbf{v}(t)\|_{\mathcal{E}} < \delta \text{ and } \|\mathbf{v}(t)\|_{\mathcal{E}} \leqslant C_{\delta_0} \left\|\mathbf{v}^0\right\|_{\mathcal{E}} \tag{41}$$

hold for the solution $\mathbf{v}(t)$ of (8) with the initial value $\mathbf{v}^0 \in \mathcal{E}^2$ as long as $\|\mathbf{v}^0\|_{\mathcal{E}} < \gamma$. Next, we show that for each $\delta \in (0, \delta_0)$ there is an η such that, if $\|\mathbf{v}^0\|_{\mathcal{E}} < \eta$, then $\|\mathbf{v}(t)\|_{\mathcal{E}} < \delta$ for all $t \geqslant 0$. Assume $C_{\delta_0} > 1$; without loss of generality, by setting $\eta = \frac{\delta}{2C_{\delta_0}}$ and assuming $\|\mathbf{v}^0\|_{\mathcal{E}} < \eta$, then $\|\mathbf{v}(T(\delta, \gamma))\|_{\mathcal{E}} < \delta/2$ by (41). Thus, the solution $\mathbf{v}$ with the initial value $\mathbf{v}(T(\delta, \gamma))$) again satisfies (41) for $t \in [T(\eta, \gamma), 2T(\eta, \gamma))$ per Propositions 4 and 5. Thus, these propositions can be applied for all $t \geqslant 0$ which proves its stability. As long as these propositions are applicable, we are able obtain more refined information about the behavior of the solution, including its boundedness in the $\|\cdot\|_0$-norm and the exponential decay in the $\|\cdot\|_\alpha$-norm; see items (3)–(5) of Theorem 1.

With an initial value $\mathbf{v}^0 \in \mathcal{E}^2$, let $\mathbf{v}(t) = \mathbf{v}(t, \mathbf{v}^0)$ be the solution of (8) in $\mathcal{E}^2$, which we have already shown to exist on at least a short time period. We now complete the proof of the nonlinear stability of the end state $\mathbf{u}_-$ by obtaining a control on solutions for all times t. For this, we need to use the following general result (see [22] Proposition 1.21):

Lemma 5 (Abstract Bootstrap Principle). *First, let I be a time interval; then, suppose that for each $T \in I$ we have two statements, a "hypothesis" $H(T)$ and a "conclusion" $C(T)$. Suppose that we can verify the following four assertions:*

(a) *(Hypothesis implies conclusion): If $H(T)$ is true for some time $T \in I$, then $C(T)$ is true for that time T.*
(b) *(Conclusion is stronger than hypothesis): If $C(T)$ is true for some time $T \in I$, then $H(T')$ is true for all $T' \in I$ in a neighborhood of T.*
(c) *(Conclusion is closed): If $T_1, T_2, \ldots$ is a sequence of times in I which converges to another time $T \in I$, and $C(T_n)$ is true for all T_n, then $C(T)$ is true.*
(d) *(Base case): Of $H(T)$ is true for at least one time $T \in I$, then $C(T)$ is true for all $T \in I$.*

We now state the main result in Theorem 1. The small constant δ_0 in the proof can be taken as $\delta_0 = \delta_2$, where δ_2 is chosen as in Proposition 6.

Theorem 1. *Let $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ with $k \geq \frac{d+1}{2}$ and consider the semilinear system (8). There exist constants $C > 0$, $\nu > 0$, and a small $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ we can find η to satisfy $0 < \eta < \delta$ such that, if $||\mathbf{v}^0||_{\mathcal{E}} \leq \eta$, the following is true for the solution $\mathbf{v}(t)$ of (8) for all $t > 0$:*

(1) $\mathbf{v}(t)$ *is defined in* $\mathcal{E}^2$
(2) $||\mathbf{v}(t)||_{\mathcal{E}} \leq \delta$
(3) $||\mathbf{v}(t)||_\alpha \leq Ce^{-\nu t}||\mathbf{v}^0||_\alpha$
(4) $||v_1(t)||_0 \leq C||\mathbf{v}^0||_{\mathcal{E}}$
(5) $||v_2(t)||_0 \leq Ce^{-\nu t}||\mathbf{v}^0||_{\mathcal{E}}$.

Proof. Let I be the time interval $[0, \infty)$.

Let $H(T)$ in Lemma 5 be the following statement. For each $0 < \delta < \delta_2$ in which δ_2 is chosen as in Proposition 6, there exists $0 < \gamma < \delta$ such that if $||\mathbf{v}^0||_{\mathcal{E}} \leq \gamma$, then $\mathbf{v}(t)$ is defined and $||\mathbf{v}(t)||_{\mathcal{E}} \leq \delta$ on the time interval $[0, T)$ for some $T = T(\delta, \gamma)$, depending on δ and γ. Thus, property (d) of the bootstrap principle is proven by Proposition 4.

Let $C(T)$ be the following statement. There exists $T > 0$ such that properties (3)–(5) in Theorem 1 hold within the time interval $[0, T)$.

Let $0 < \gamma_1 < \delta < \delta_2$ and let $\gamma = C^{-1}\gamma_1$, where C is a constant satisfying $C > \max\{1, K_1, C_1\}$ with K_1 and C_1 as in Propositions 5 and 6.

Let $\mathbf{v}^0 \in \mathcal{E}^2$ with $||\mathbf{v}^0||_{\mathcal{E}} \leq \gamma$ be the initial value of (8). Now, $\gamma \leq \gamma_1 < \delta$. Choose ν as in Proposition 1; per Propositions 5 and 6, items (3), (4), and (5) hold for $0 < t \leq T(\delta, \gamma)$. Thus, property (a) of the bootstrap principle is proven.

Property (c) holds by the continuity of $\mathbf{v}(t)$.

Now, we need to prove property (b) of the bootstrap principle. Let $\mathbf{v}^0 \in \mathcal{E}^2$ with $||\mathbf{v}^0||_{\mathcal{E}} \leq \gamma$; for any $t \in (0, T)$, inequilities (3), (4), and (5) hold. Thus, by continuity of $\mathbf{v}(t)$, we can conclude that

$$||\mathbf{v}(T, \mathbf{v}^0)||_{\mathcal{E}} \leq C||\mathbf{v}^0||_{\mathcal{E}} \leq C\gamma = \gamma_1. \tag{42}$$

If we take $\mathbf{v}^1 = \mathbf{v}(T, \mathbf{v}^0)$ as an initial value of system (8), it satisfies $||\mathbf{v}^1||_{\mathcal{E}} \leq \gamma_1 < \delta$; by applying Proposition 4 again, there exists $T(\delta, \gamma_1) > 0$ such that for all $t \in (T, T + T(\delta, \gamma_1))$ we have

$$||\mathbf{v}(T + t, \mathbf{v}^0)||_{\mathcal{E}} = ||\mathbf{v}(t, \mathbf{v}^1)||_{\mathcal{E}} \leq \delta. \tag{43}$$

Then, $H(T')$ is true for $T' = T + T(\delta, \gamma_1)$, and property (b) is proven.

Thus, we finish the proof of Theorem 1 using the bootstrap principle. □

4. Stability of the End States for a General System

In this section, we study a steady state solution $\mathbf{u}_-$ to (4) with $f(\mathbf{u}_-) = 0$ and its perturbation depending on the spatial variable $\mathbf{x} \in \mathbb{R}^d$.

Without loss of generality, we take $\mathbf{u}_- = 0$. Information about the stability of the zero solution is encoded in the spectrum of the operator obtained by linearizing (4) with respect to zero:

$$\mathbf{u}_t = D\Delta_{\mathbf{x}} u + c\partial_z \mathbf{u} + \partial_{\mathbf{u}} f(0)\mathbf{u} =: L\mathbf{u}, \tag{44}$$

where $\partial_{\mathbf{u}}$ is the differential with respect to $\mathbf{u}$.

Let $\mathcal{E}_0$ be the Sobolev spaces $H^k(\mathbb{R}^d)$, and define the weight function

$$\gamma_\alpha(z, x_2, \ldots, x_d) = e^{\alpha z}$$

and the spaces $\mathcal{E}_\alpha$ and $\mathcal{E} = \mathcal{E}_0 \cap \mathcal{E}_\alpha$ as before. Analogously to the model problem discussed in Section 3, we use $\mathcal{L}$ to denote the operator defined on $\mathcal{E}_0^2$ provided by the map $u \to Lu$, with the domain $u \in H^{k+2}(\mathbb{R}^d)^2$. We use $\mathcal{L}_\alpha$ to denote the operator defined on $\mathcal{E}_\alpha^2$ as provided by $u \to Lu$, with the domain being the set of u, where $\gamma_\alpha u \in H^{k+2}(\mathbb{R}^d)^2$. Throughout, we impose the following assumptions on $f(\cdot)$ in (4).

Hypothesis 1.

(a) *In appropriate variables* $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_1 \in \mathbb{R}^{n_1}, \mathbf{u}_2 \in \mathbb{R}^{n_2}, n_1 + n_2 = n$, *we assume that for some constant* $n_1 \times n_1$ *matrix* A_1 *we have*

$$f(\mathbf{u}_1, 0) = (A_1\mathbf{u}_1, 0)^T.$$

(b) *The function* f *is* C^{k+3} *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$.

If Hypothesis 1 holds, then $f(\mathbf{u}_\pm) = 0$ and

$$\begin{aligned} f(\mathbf{u}_1, \mathbf{u}_2) &= f(\mathbf{u}_1, 0) + f(\mathbf{u}_1, \mathbf{u}_2) - f(\mathbf{u}_1, 0) \\ &= \begin{pmatrix} A_1\mathbf{u}_1 \\ 0 \end{pmatrix} + \int_0^1 \partial_{\mathbf{u}_2} f(\mathbf{u}_1, t\mathbf{u}_2) dt \mathbf{u}_2 \\ &= \begin{pmatrix} A_1\mathbf{u}_1 + \tilde{f}_1(\mathbf{u}_1, \mathbf{u}_2)\mathbf{u}_2 \\ \tilde{f}_2(\mathbf{u}_1, \mathbf{u}_2)\mathbf{u}_2 \end{pmatrix}, \end{aligned}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are some matrix-valued functions of size $n_1 \times n_2$ and $n_2 \times n_2$, respectively. Then, we can write

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad f(\mathbf{u}) = \begin{pmatrix} f_1(\mathbf{u}_1, \mathbf{u}_2) \\ f_2(\mathbf{u}_1, \mathbf{u}_2) \end{pmatrix},$$

where each D_i is a nonnegative diagonal matrix of size $n_i \times n_i$ and $f_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ for $i = 1, 2$. Equation (4) now reads as follows:

$$\partial_t \mathbf{u}_1 = D_1 \Delta_x \mathbf{u}_1 + c\partial_z \mathbf{u}_1 + f_1(\mathbf{u}_1, \mathbf{u}_2), \tag{45}$$

$$\partial_t \mathbf{u}_2 = D_2 \Delta_x \mathbf{u}_2 + c\partial_z \mathbf{u}_2 + f_2(\mathbf{u}_1, \mathbf{u}_2). \tag{46}$$

If we linearize (46) at $(0,0)$, then the constant–coefficient linear equation depends only on u_2, as $f(0,0) = 0$ per Hypothesis 1 (a) can be obtained as follows:

$$\begin{aligned} \partial_t \mathbf{u}_2 &= D_2 \Delta_{\mathbf{x}} \mathbf{u}_2 + c\partial_z \mathbf{u}_2 + \partial_{\mathbf{u}_1} f_2(0,0)\mathbf{u}_1 + \partial_{\mathbf{u}_2} f_2(0,0)\mathbf{u}_2 \\ &= D_2 \Delta_{\mathbf{x}} \mathbf{u}_2 + c\partial_z \mathbf{u}_2 + \partial_{\mathbf{u}_2} f_2(0,0)\mathbf{u}_2. \end{aligned} \tag{47}$$

We denote by $L^{(2)}\mathbf{u}_2$ the right-hand side of (47) and let $\mathcal{L}^{(2)}$ be the operator defined on $H^k(\mathbb{R}^d)^{n_2}$ and provided by $\mathbf{u} \to L^{(2)}\mathbf{u}$ with the domain $\mathbf{u} \in H^{k+2}(\mathbb{R}^d)^{n_2}$.

In addition, we linearize (45) at $(0,0)$; per by Hypothesis 1 (a), the respective constant–coefficient linear equation reads

$$\begin{aligned} \partial_t \mathbf{u}_1 &= D_1 \Delta_{\mathbf{x}} \mathbf{u}_1 + c\partial_z \mathbf{u}_1 + \partial_{\mathbf{u}_1} f_1(0,0)\mathbf{u}_1 + \partial_{\mathbf{u}_2} f_1(0,0)\mathbf{u}_2 \\ &= D_1 \Delta_{\mathbf{x}} \mathbf{u}_1 + c\partial_z \mathbf{u}_1 + A_1 \mathbf{u}_1 + \partial_{\mathbf{u}_2} f_1(0,0)\mathbf{u}_2. \end{aligned} \tag{48}$$

We denote $L^{(1)}\mathbf{u}_1 = D_1 \Delta_{\mathbf{x}} \mathbf{u}_1 + c\partial_z \mathbf{u}_1 + A_1 \mathbf{u}_1$; thus, $\partial_t \mathbf{u}_1 = L^{(1)}\mathbf{u}_1 + \partial_{\mathbf{u}_2} f_1(0,0)\mathbf{u}_2$. Let $\mathcal{L}^{(1)}$ be the operator defined on $H^k(\mathbb{R}^d)^{n_1}$, provided by $\mathbf{u} \to L^{(1)}\mathbf{u}$, with the domain $\mathbf{u} \in H^{k+2}(\mathbb{R}^d)^{n_1}$.

With the additional assumptions listed below, we show that the perturbations of the left end state $\mathbf{u}_-$ that are initially small in both the unweighted norm and weighted norm remain small in the unweighted norm and decay exponentially in the weighted norm. In addition, the $\mathbf{u}_2$-component of the perturbation decays exponentially in the unweighted norm. Below, we use the following hypotheses about the spectrum of $\mathcal{L}$.

Hypothesis 2. *In addition to Hypothesis 1, we assume that there exists a constant $\alpha > 0$ such that $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\alpha)\} < 0$ on $(L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}))^n$.*

As in Section 3.1, let $y = (x_2, \dots, x_d) \in \mathbb{R}^{d-1}$ such that $\mathbf{x} = (z, y) \in \mathbb{R}^d$, denote $L_{1,\alpha} = D\partial_{zz} + c\partial_z + \partial_{\mathbf{u}} f(0)\mathbf{u}$ and $\Delta_y = \partial^2_{x_2} + \cdots + \partial^2_{x_d}$, and define the linear operators $\mathcal{L}_{1,\alpha} : H^k_\alpha(\mathbb{R})^n \to H^k_\alpha(\mathbb{R})^n$, where $\operatorname{dom}(\mathcal{L}_{1,\alpha}) = H^{k+2}_\alpha(\mathbb{R})^n \subset H^k_\alpha(\mathbb{R})^n$ and $\Delta_y : H^k(\mathbb{R}^{d-1})^n \to H^k(\mathbb{R}^{d-1})^n$, where $\operatorname{dom}(\Delta_y) = H^{k+2}(\mathbb{R}^{d-1})^n \subset H^k(\mathbb{R}^{d-1})$. Then, the operator $\mathcal{L}_\alpha$ on the space $H^k_\alpha(\mathbb{R}^d)^n$ can be represented as

$$\mathcal{L}_\alpha = \mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y.$$

Hypothesis 2 holds if there exists a constant $\alpha > 0$ such that $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_{1,\alpha})\} < 0$ on $H^k_\alpha(\mathbb{R})^n$. Indeed, following Remark 3, we have

$$\operatorname{Sp}\Big(\mathcal{L}_{1,\alpha} \otimes I_{H^k(\mathbb{R}^{d-1})} + I_{H^k_\alpha(\mathbb{R})} \otimes \Delta_y\Big) = \operatorname{Sp}(\mathcal{L}_{1,\alpha}) + \operatorname{Sp}(\Delta_y)$$

Note that the spectra of $\mathcal{L}_{1,\alpha}$ on $L^2_\alpha(\mathbb{R}^d)^n$ and $H^k_\alpha(\mathbb{R}^d)^n$ are equal, similar to Lemma 6; thus, it is apparent that if $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_{1,\alpha})\} < 0$ on $L^2_\alpha(\mathbb{R})^n$, then Hypothesis 2 is satisfied for any $H^k_\alpha(\mathbb{R}^d)^n$.

Hypothesis 3. *In addition to Hypothesis 2, we assume the following:*

(1) The operator $\mathcal{L}^{(1)}$ generates a bounded semigroup on the spaces $L^2(\mathbb{R}^d)^{n_1}$ and $H^k(\mathbb{R}^d)^{n_1}$.

(2) *The operator* $\mathcal{L}^{(2)}$ *satisfies* $\sup\left\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}\left(\mathcal{L}^{(2)}\right)\right\} < 0$ *on* $L^2(\mathbb{R}^d)^{n_2}$ *and* $H^k(\mathbb{R}^d)^{n_2}$.

Note that we have used the following lemmas in stating these hypotheses, by analogy to Lemma 1 in Section 3.1.

Lemma 6. *The linear operator* $\mathcal{L}$ *associated with* L *in* (44) *has the same spectrum on* $L^2(\mathbb{R}^d)^n$ *and* $H^k(\mathbb{R}^d)^n$*, and the linear operators* $\mathcal{L}^{(i)}$ *associated with* $L^{(i)}$ *in* (48) *and* (47) *have the same spectra on* $L^2(\mathbb{R}^d)^{n_i}$ *and* $H^k(\mathbb{R}^d)^{n_i}$ *for* $i = 1, 2$*; similarly, the linear operator* $\mathcal{L}_\alpha$ *has the same spectrum on both* $(L^2_\alpha(\mathbb{R}) \otimes L^2(\mathbb{R}^{d-1}))^n$ *and* $(H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1}))^n$.

Proof. Because $\mathcal{L}$ is associated with the constant–coefficient differential expression L, we can use the same proof as in Lemma 1. □

We now rewrite Equation (4) for the perturbation $\mathbf{v}(t, \mathbf{x})$ of the end state $\mathbf{u}_- = 0$ in a form amenable to the subsequent analysis. We seek a solution to (4) of the form $\mathbf{u}(t, \mathbf{x}) = \mathbf{u}_- + \mathbf{v}(t, \mathbf{x})$. Using this notation, $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ satisfies

$$\mathbf{v}_t = D\Delta_{\mathbf{x}}\mathbf{v} + c\partial_z\mathbf{v} + \partial_{\mathbf{u}} f(0)\mathbf{v} + f(\mathbf{v}) - f(0) - \partial_{\mathbf{u}} f(0)\mathbf{v}. \tag{49}$$

Note that

$$f(\mathbf{v}) - f(0) - \partial_{\mathbf{u}} f(0)\mathbf{v} = \int_0^1 (\partial_{\mathbf{u}} f(t\mathbf{v}) - \partial_{\mathbf{u}} f(0))dt\mathbf{v}.$$

We define

$$N(\mathbf{u}) = \int_0^1 (\partial_{\mathbf{u}} f(t\mathbf{v}) - \partial_{\mathbf{u}} f(0))dt, \tag{50}$$

as an $n \times n$ matrix-valued function of $\mathbf{v}$; note that $N(\mathbf{v})\mathbf{v} \in \mathbb{R}^n$ for any $\mathbf{v} \in \mathbb{R}^n$. Using (50), we can rewrite (49) as

$$\mathbf{v}_t = L\mathbf{v} + N(\mathbf{v})\mathbf{v}. \tag{51}$$

This is the semilinear equation for the perturbation we examine here. Throughout the rest of this section, we assume $k \geqslant \left[\frac{d+1}{2}\right]$ for $\mathcal{E}_0 = H^k(\mathbb{R}^d)$ and $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$.

Proposition 7. *Assuming that Hypotheses 1–3 hold, the following are true. (1) There exists* $\alpha > 0$ *such that on the weighted space* $\mathcal{E}^n_\alpha$*, the spectrum of* $\mathcal{L}_\alpha$ *is bounded away from the imaginary axis* $\sup\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}(\mathcal{L}_\alpha)\} < -\nu$ *for some* $\nu > 0$*. In addition, there exists* $K > 0$ *such that*

$$\left\|e^{t\mathcal{L}_\alpha}\right\|_{\mathcal{E}^n_\alpha \to \mathcal{E}^n_\alpha} \leqslant Ke^{-\nu t} \quad \text{for all} \quad t \geqslant 0.$$

(2) On the unweighted space $\mathcal{E}^{n_2}_0$*, we have* $\sup\left\{\operatorname{Re}\lambda : \lambda \in \operatorname{Sp}\left(\mathcal{L}^{(2)}\right)\right\} < -\rho$ *for some* $\rho > 0$*, and there exists* $K > 0$ *such that* $\left\|e^{t\mathcal{L}^{(2)}}\right\|_{\mathcal{E}^{n_2}_0 \to \mathcal{E}^{n_2}_0} \leqslant Ke^{-\rho t}$ *for all* $t \geqslant 0$.

Proof. Statement (1) holds by Hypothesis 2 and Lemma 6, while Statement (2) follows from Hypothesis 3 and Lemma 6. □

The above Proposition 7 provides the spectral stability of the linear operator in the semilinear system (51). We next estimate the local Lipschitz property for the nonlinear terms $N(\mathbf{v})\mathbf{v}$, as in (51) for the weighted and unweighted norms.

Proposition 8. *Assume* $k \geqslant \frac{d+1}{2}$ *and let* $\mathcal{E}_0 = H^k(\mathbb{R}^d)$*; given* $f \in C^{k+3}(\mathbb{R}^n; \mathbb{R}^n)$*, consider the nonlinearity* $N(\mathbf{v})$ *defined in* (50)*. Then, we have the following:*

(1) *If* $\mathbf{v} \in \mathcal{E}^n$*, then* $N(\mathbf{v})\mathbf{v} \in \mathcal{E}^n_\alpha$*, and on any bounded neighborhood with the form* $\{\mathbf{v} : \|\mathbf{v}\|_{\mathcal{E}} \leqslant K\}$ *there is a constant* $C_K > 0$ *such that* $\|N(\mathbf{v})\mathbf{v}\|_\alpha \leqslant C_K\|\mathbf{v}\|_0\|\mathbf{v}\|_\alpha$.

(2) *If $\mathbf{v} \in \mathcal{E}^n$, then $N(\mathbf{v})\mathbf{v} \in \mathcal{E}_0^n$, and on any bounded neighborhood with the form $\{\mathbf{v} : \|\mathbf{v}\|_0 \leqslant K\}$ there is a constant $C_K > 0$ such that $\|N(\mathbf{v})\mathbf{v}\|_0 \leqslant C_K \|\mathbf{v}\|_0^2$.*

(3) *The formula $\mathbf{v} \mapsto N(\mathbf{v})\mathbf{v}$ defines a mapping from $\mathcal{E}^n$ to $\mathcal{E}^n$ that is locally Lipschitz on any bounded neighborhood with the form $\{\mathbf{v} : \|\mathbf{v}\|_{\mathcal{E}} \leqslant K\}$ in $\mathcal{E}^n$.*

In this proof, we refer to Lemma 4 for the components of $N(\mathbf{v})\mathbf{v}$ by dropping q from the lemma. Note that

$$N(\mathbf{v}) = \int_0^1 (\partial_u f(t\mathbf{v}) - \partial_u f(0))dt = \int_0^1 \left(\int_0^1 \partial_{u^2} f(st\mathbf{v})ds \right) tvdt.$$

By applying Lemma 4 to the components of the vector under the integral, the mapping $\mathbf{v} \mapsto N(\mathbf{v})$ is locally Lipschitz on sets with the form $\{\mathbf{v} : \|\mathbf{v}\|_0 \leqslant K\}$, satisfying

$$\|N(\mathbf{v})\|_0 \leqslant C_K \|\mathbf{v}\|_0. \tag{52}$$

Thus, per Lemma 3 (1) and (52), we conclude that the nonlinearity $N(\mathbf{v})\mathbf{v}$ satisfies

$$\|N(\mathbf{v})\mathbf{v}\|_0 \leqslant \|N(\mathbf{v})\|_0 \|\mathbf{v}\|_0 \leqslant C_K \|\mathbf{v}\|_0 \|\mathbf{v}\|_0,$$

while per Lemma 3 (2) and (52), it satisfies

$$\|N(\mathbf{v})\mathbf{v}\|_\alpha = \|\gamma_\alpha N(\mathbf{v})\mathbf{v}\|_0 \leqslant \|N(\mathbf{v})\|_0 \|\gamma_\alpha \mathbf{v}\|_0 \leqslant C_K \|\mathbf{v}\|_0 \|\mathbf{v}\|_\alpha,$$

as well, thus proving (1) and (2). Next, we use the definition of $\|\,\|_{\mathcal{E}}$ and infer

$$\begin{aligned} \|N(\mathbf{v})\mathbf{v}\|_{\mathcal{E}} &= \max\{\|N(\mathbf{v})\mathbf{v}\|_0, \|N(\mathbf{v})\mathbf{v}\|_\alpha\} \\ &\leqslant \max\{C_K \|\mathbf{v}\|_0 \|\mathbf{v}\|_0, C_K \|\mathbf{v}\|_0 \|\mathbf{v}\|_\alpha\} \\ &\leqslant C_K \|\mathbf{v}\|_{\mathcal{E}} \|\mathbf{v}\|_{\mathcal{E}}. \end{aligned}$$

With the information that we have now obtained, the spectrum of the linear operator of system (51) is stable in the weighted space and the nonlinear terms of system (51) under the weighted norm satisfy certain locally Lipchitz conditions. Next, we proceed as in the proof of Propostions 4–6 in Section 3 and use similar Bootstrap arguments as those used in the proof of Theorem 1 to finally obtain the following stability result.

Theorem 2. *With an initial value $\mathbf{v}^0 \in \mathcal{E}^n$, let $\mathbf{v}(t) = v(t, \mathbf{v}^0)$ be the solution of (51) in $\mathcal{E}^n$ with $\mathbf{v}(0) = \mathbf{v}^0$, and let $k \geqslant \frac{d+1}{2}, \mathcal{E}_0 = H^k(\mathbb{R}^d)$ and $\mathcal{E}_\alpha = H^k_\alpha(\mathbb{R}) \otimes H^k(\mathbb{R}^{d-1})$. Assuming Hypotheses 1–3, there exist constants $C > 0, \nu > 0$ and a small $\delta_0 > 0$ such that for each $0 < \delta < \delta_0$ we can find $\eta > 0$ such that, if $\|\mathbf{v}^0\|_{\mathcal{E}} \leqslant \eta$, then the following is true for all $t > 0$:*

- *$\mathbf{v}(t)$ is defined in $\mathcal{E}^n$*
- *$\|\mathbf{v}(t)\|_{\mathcal{E}} \leqslant \delta$*
- *$\|\mathbf{v}(t)\|_\alpha \leqslant Ce^{-\nu t}\|\mathbf{v}^0\|_\alpha$*
- *$\|\mathbf{v}_1(t)\|_0 \leqslant C\|\mathbf{v}^0\|_{\mathcal{E}}$*
- *$\|\mathbf{v}_2(t)\|_0 \leqslant Ce^{-\nu t}\|\mathbf{v}^0\|_{\mathcal{E}}$.*

As the proof is identical to the proof of Theorem 1, we do not restate it here.

5. Conclusions and Future Work

In this paper, we have studied a class of reaction–diffusion systems usually associated with combustion problems. In particular, they are characterized by a special product triangle structure, that is, the linear operator obtained after linearization of the system with respect to the traveling front has a triangular structure (25) and the nonlinear reaction terms have a product structure. This structure is caused by the strong dependence of the reaction rate on the temperature, which is displayed as a cut-off from the source in terms of mathematical expression. Examples of other systems that possess this type of structure include the exothermic-endothermic chemical reaction

$$\begin{aligned} \partial_t y_1 &= \Delta_{\mathbf{x}} y_1 + y_2 f_2(y_1) - \sigma y_3 f_3(y_1) \\ \partial_t y_2 &= d_2 \Delta_{\mathbf{x}} y_2 - y_2 f_2(y_1) \\ \partial_t y_3 &= d_3 \Delta_{\mathbf{x}} y_3 - \tau y_3 f_3(y_1) \end{aligned}$$

Here, y_1 is the temperature, y_2 is the quantity of exothermic reactant, and y_3 is the quantity of endothermic reactant. The parameters σ and τ are positive, and there are positive constants a_i and b_i such that $f_i(y_1) = a_i e^{-\frac{b_i}{y_1}}$ for $y_1 > 0$ and $f_i(y_1) = 0$ for $y_1 \leqslant 0$, as well as the gasless combustion

$$\partial_t u = \Delta_{\mathbf{x}} u + v g(u), \quad \partial_t v = -\beta v g(u),$$

where $g(u) = e^{-\frac{1}{u}}$ if $u > 0$ and $g(u) = 0$ if $u \leqslant 0$. In this system, u is the temperature, v is the concentration of unburned fuel, g is the unit reaction rate, and $\beta > 0$ is a constant parameter.

For a reaction–diffusion system with this structure, we show that if the spectrum of the linear operator projected in one-dimensional space is touching the imaginary axis, a weight function and weighted space can be used to shift the spectrum of the linear operator to the left to obtain the spectral stability of the operator. On the other hand, we show that the nonlinear reaction term with the product form has the local Lipschitz property in the constructed weighted space. By combining these facts, the stability of the steady-state solution of the planar front can be obtained.

However, there are several problems involving this same subject that remain unsolved for the time being. For example, the linear operator obtained by linearizing the system with respect to the planar front has isolated singularities, and each of these isolated singularities extends an infinite semiline in the multidimensional space, as discussed in Remark 3. This which leads us to presuppose, as in [15], that the diffusion coefficients of different variables of the system are identical; interested readers may refer to [15] (Proposition 3.1) for a detailed discussion.

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