MDPI

Article

Optimization of Fire Brigade Deployment by Means of Mathematical Programming

Bohuš Leitner [1,*] , **Michal Ballay** [1] , **Marek Kvet** [2,*] and **Michal Kvet** [2]

[1] Faculty of Security Engineering, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia
[2] Faculty of Management Science and Informatics, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia
[*] Correspondence: bohus.leitner@uniza.sk (B.L.); marek.kvet@uniza.sk (M.K.)

Abstract: The content of this research paper is focused on the applications of selected operation research methods in the field of rescue services. The first theoretical part aims at identifying the most important aspects of the real system, which should be taken into account whenever a rescue system is to be redesigned or optimized. It also contains an in-depth description of system performance. Based on the found criteria, a mathematical model for a new system design is proposed. The second part of this paper is practically oriented. A short case study performed with real data on the rescue service system in Slovakia is reported herein, and the obtained results are compared with current fire brigade deployment. We concentrate on improving the most important criteria of the real system. If the suggested mathematical model proves to be hard to solve with common optimization tools, we will develop an efficient solving algorithm. From a general point of view, the main scientific goal of this research article is to summarize current trends in location science, which could be useful for the optimization of fire brigade deployment. Since the strategic decisions in location service centers to some nodes of the network are usually made for long periods, different properties and possible failures are discussed to be considered with the mathematical models and associated solving tools.

Keywords: location analysis; fire brigade deployment; mathematical modeling; service accessibility improvement

1. Introduction

One of the most important functions of every good human society is to protect people and minimize potential dangers caused by unexpected threats to life and health from various natural or industrial sources [1–8]. This activity is largely handled by the Integrated Rescue System (IRS), which also includes the Fire and Rescue Service. Such a system is also operated in Slovakia [9–13].

Firefighting units in every country of the world form a system that protects the health and lives of citizens and cannot be based on chance or only on the will and willingness to help [14–16]. From a social point of view, every country tries to promote the flawless functioning of this system and to find the optimal level of financial costs for its performance. Obviously, all fire departments cannot be equipped or trained equally, and it makes no sense for them all to have the same response time.

The state helps citizens in fighting fires and carrying out rescue work in the case of natural disasters and other extraordinary events by sending Fire Protection Units (FPUs). Other entities also have the obligation to establish fire protection units, e.g., significant industrial enterprises with a higher degree of fire risk, the Armed Forces of the Slovak Republic, and also the local government in the form of the village's Voluntary Fire Brigades. It is therefore possible to create a system of mutual assistance, which creates a prerequisite for the provision of assistance to citizens within a time limit of up to 15 min. Such a system of providing assistance within a specified time limit and with a predetermined number of

fire protection units is defined in the conditions of the Slovak Republic in the Fire Protection Act (314/2001 Coll.) and the Decree of the Ministry of the Interior of the Slovak Republic on firefighting units (611/2006 Z.z.).

The internal organization and equipment of fire protection units, including the deployment of individual types and categories of fire protection units, must be determined so that the territory of a municipality is secured with the required number of forces and resources according to the degree of danger. From this point of view, the deployment of fire brigades plays an essential role in the system's performance. Each type of fire protection unit has its operational value (operational equivalent). This value indicates the ability of the fire protection unit to carry out activities involving fighting fires and rescue work during natural disasters and other extraordinary events. The operational value of a fire protection unit consists of the departure time after the alarm is declared and the territorial jurisdiction. The departure time is the period from the declaration of the alarm for the designated forces and means of fire protection to their departure from the place of their deployment. The law for various types of fire brigades establishes this period. The territorial coverage is the optimal distance for a certain type of fire protection unit to reach the place of intervention, expressed as the travel time in minutes, which defines the territory of its standard operation, the so-called intervention circuit. Obviously, it can be generally assumed that the travel time is proportional to the distance needed to be traveled within the associated transportation network, i.e., roads [9–13].

Without any doubt, finding the optimal deployment of forces and resources is a difficult challenge for all decision-makers responsible for the quality of the service provided to citizens affected by an emergency. In this paper, we want to point out that operations research may serve as a core scientific field to support responsible authorities when searching for a new deployment of fire brigades in the network covering the served geographical area [2,7,9,15]. Since the problem of locating forces, resources, and employees can be formulated in mutual ways, making use of mathematical models and/or integer linear programming, we concentrate herein on a brief summary of current trends in location science that are applicable to the family of rescue services [1,5–7,11,12,14–17]. Obviously, we provide the readers with plenty of references to recently published papers, which contain the results of performed numerical experiments with the dataset of the Emergency Medical Service (EMS) system being operated in Slovak self-governing regions.

This research paper is structured in the following way: After the introduction section, there are explained basic modeling concepts with response time minimization and coverage criteria [17–19]. The mathematical models are discussed in detail together with possible modifications and reformulations enabling their better solvability. Basic models are accompanied by a small computational study, in which we emphasize the most important points and (dis)advantages of particular approaches. The second part of the paper is focused on theoretical advanced challenges. The proposed mathematical models are more complicated, and they try to consider various aspects of the real world that may significantly affect the system's performance, and from this point of view, they should not be abstracted when formulating models for associated location problems. Herein, we mention robustness not only as a system's resistance to difficulties in the transportation network [6,8,16] but also as the concept of so-called reengineering, which follows from the existing system and limits the number of performed changes in current fire brigade deployment [20,21]. It may be useful for the public to accept new system designs better and more easily. Finally, we present herein the possibility of managing the problem of searching for new modes of fire brigade deployment using a bi-criteria model, which brings several new challenges and requires the development of new approaches to solving [22–26]. Many of the following sections are accompanied by a small illustrative example in a figure to explain the concepts being discussed better. We hope that such small practical examples could help to make the sections more accessible to those readers who may not have a deep background in fire brigade deployment or optimization.

To sum up the main contribution of this paper, it is aimed at various kinds of optimization approaches, the goal of which is to find an appropriate mode of fire brigade deployment, which would benefit from considering several aspects of the real world in the decision-making process. This means that we provide the readers of this article with a summary of current trends in location science for emergency services. Due to the lack of real-world data, several concepts are discussed only in a theoretical way. Nevertheless, future research in this science could emphasize the real impact of optimization approaches on real system performance efficiency and service accessibility for clients.

2. Basic Modeling Concepts

This section addresses the problem of the optimal deployment of fire protection units using operations research tools. Because this problem can be formulated in many different simple or advanced ways, the most common objectives that are optimized are discussed herein.

The basic and simultaneously the most commonly used modeling concept follows the fact that fire brigades, firefighters, and available equipment are limited. Financial budget is also one of the biggest restrictions, which needs to be considered. It is natural that a fire brigade cannot be established wherever one would like it to be. The simplest modeling concept is based on the task of selecting a given number of fire brigades from a set of candidate locations in order to optimize a given quality criterion. The optimized objective function usually takes the form of total or average disutility expressed as the social or financial costs proportional to the distances fire protection units have to go to reach the demand points in which the associated services need to be provided. The response activity of firefighting units in the Slovak Republic consists not only of fighting fires but also of carrying out other interventions to save the lives and health of people, property, and the environment. Since the 1990s in the last century, the development of intervention activities of fire brigades has been characterized by a significant increase in activities aimed at carrying out rescue work in natural disasters, road traffic accidents, and other situations, as well as an increase in the number of technical interventions of various kinds. Let us return to the optimized quality criteria. The duration between an emergency call for help and the firefighters' departure to the scene of the extraordinary incident can also be used to describe the aforementioned inutility. Of course, perceived disutility decreases with increasing proximity to a fire station.

Based on the above-mentioned theoretical starting points, the first modeling concept follows the weighted *p*-median problem with thousands of potential service recipients [1–3,9,11,12,18–21,23–26]. The cardinality of the set of candidates, in which a fire station could be established, may take the value of several hundred or thousands as well. Thus, the mathematical model of the problem with real-world input data represents a huge challenge, because the associated solving process may run into the computational limits of available common hardware and software tools. Therefore, it is interesting not only from a modeling point of view but also from the viewpoint of efficient algorithm development. Let us focus now on the mathematical formulation of the problem.

The solving process of the standard weighted *p*-median problem consists of searching for the best selection of exactly *p* components (*p* is a positive integer number) from a given finite list of candidates. The mentioned selection must be performed in such a way that the optimization criterion takes its optimal value. The used criterion may take several forms, but the most common one can be expressed as the average distance between system users and the nearest fire station. To put this another way, the common modeling technique takes into account the so-called average service recipient.

In order to formulate the problem mathematically in the form (1)–(6) by means of linear programming, the following notations have to be introduced. Let *I* be the set of candidates for placing a fire station or another similar resource. The set *I* is formed by a list of network nodes, which fulfill the standards for establishing a fire station. As we know, the fire station requires a special building and meeting different technical features

and other aspects. Similarly, let *J* represent the list of service recipients. This set is usually formed by all the inhabited nodes of the associated transportation network or by those points in which the fire risk is high, i.e., industrial objects. Each element of *J* is weighted with a non-negative (but not necessarily integer) coefficient b_j. This coefficient may express the number of system users aggregated at *j* or it can estimate the frequency of demands toward the fire station from the dwelling place *j*. From the modeling point of view, the concrete interpretation of this weighting coefficient does not matter. The disutility for a client placed at *j* coming from the fire station candidate *i* is prompted as a non-negative d_{ij}. Although integer values have been shown to have some advantages, any value of d_{ij} does not necessarily need to be an integer. The problem described above is known under the notion of the weighted *p*-median problem, which has received much attention in the literature, particularly from the perspective of formulating solutions and efficient algorithms for large instances [1–3,9,11,12,18–21,23–26].

In the usual formulation of the weighted *p*-median problem, the strategically important decision of the selection of at most *p* elements (sometimes exactly *p* elements are required) from the candidate list *I* has to be made in order to optimize the given criterion. The aforementioned strategic decision of a fire station establishment may be formally modeled with a binary variable $y_i \in \{0, 1\}$ equaling 1 if a fire protection unit is to be placed at $i \in I$. To finalize the formulation of the associated model, the allocation variables $z_{ij} \in \{0, 1\}$ are introduced for each element $i \in I$ and for each node $j \in J$ to assign an affected service recipient (the entire aggregated community) located in *j* to a fire station established at the candidate item *i* with a value of 1.

To complete the model correctly, the variables y_i and z_{ij} must satisfy the given rules. The expressions (1)–(6) represent the basic model of the problem of minimizing the average distance from users to the nearest fire stations.

$$Minimize \ \frac{\sum\limits_{i \in I} \sum\limits_{j \in J} b_j d_{ij} z_{ij}}{\sum\limits_{j \in J} b_j} \tag{1}$$

$$Subject\ to: \sum_{i \in I} z_{ij} = 1 \ for\ j \in J \tag{2}$$

$$\sum_{i \in I} y_i = p \tag{3}$$

$$z_{ij} \leq y_i \ for\ i \in I, j \in J \tag{4}$$

$$y_i \in \{0, 1\} \ for\ i \in I \tag{5}$$

$$z_{ij} \in \{0, 1\} \ for\ i \in I, j \in J \tag{6}$$

Despite the fact that the above-formulated model of the weighted *p*-median problem is very well known, let us briefly recall the sense of the above expressions. The minimized objective function (1) represents the average distance from service recipients to their closest FPU. The allocation constraints (2) ensure that each served network node *j* is assigned to a located FPU. Such an assignment is achieved in combination with the link-up constraints (4). This way, the model provides the decision-makers as well as the territorial jurisdiction. Formula (3) keeps the highest possible number of located FPUs. The mandatory constraints (5) and (6) ensure the correct values of the decision variables.

Even though the weighted *p*-median problem seems easy to understand and implement in a common optimization environment, it may bring several difficulties if the solved instance acquires too big a size. Considering the fact that the model contains two big matrices (input data and decision variables), the associated branch and bound method requires a large amount of memory space. If the set of candidates contains several thousands of elements, then the usage of the model (1)–(6) becomes questionable. On the other hand,

practical problem instances have led to deeper analyses, which have resulted in a new way of problem modeling. Let us specify this in more detail.

If one looks at the model (1)–(6), the most important starting point for the new model consists of the constraints (2). Note that for each $j \in J$, there is only exactly one variable z_{ij} taking the value of one. This means that each supplied network node is associated with only one established fire station. From the standpoint of evaluating the objective function (1), it is not crucial to know which FPU is closest to network node j because the distance, not the FPU, is what matters the most. If we know the list of located FPUs, we can easily identify the nearest one for any served node. Thus, the territorial jurisdiction may be computed secondarily, and it is not necessary to model it with allocation variables. The so-called radial formulation, which was first published in [27–29] and revised in [30,31], provides the remedy for the aforementioned shortcoming of the location-allocation model (1)–(6).

The radial approach is built on the idea of excluding the assignment of served nodes to established FPUs, and it is based on a specific expression of the distances from users to their nearest fire stations. To formulate the radial model, let the symbol m denote the highest value in the matrix $\{d_{ij}\}$, i.e., $m = \max\{d_{ij} : i \in I, j \in J\}$. Let us assume that the distances in the matrix are integers. Otherwise, the covering approach could be easily adjusted [31]. For each served network node $j \in J$ and for each integer value $v = 0, 1 \dots m - 1$, we introduce a binary variable $x_{jv} \in \{0,1\}$. This auxiliary variable equals one if the distance d_{j*} from the served network node $j \in J$ to the closest FPU is greater than v; otherwise, it takes the value of zero. After that, the distance d_{j*} for each $j \in J$ can be reformulated as (7).

$$d_{j*} = \sum_{v=0}^{m-1} x_{jv} \; for \, j \in J \tag{7}$$

Similar to the common set covering problem, a matrix $\{a^s{}_{ij}\}$ has to be defined according to the following expression (8).

$$a^v{}_{ij} = \begin{cases} 1 & if \, d_{ij} \leq v \\ 0 & otherwise \end{cases} \; for \, i \in I, j \in J, v = 0, 1, \dots m - 1 \tag{8}$$

After introducing all input data structures, the problem's radial model can be expressed as (9)–(13).

$$Minimize \quad \frac{\sum\limits_{j \in J} b_j \sum\limits_{v=0}^{m-1} x_{jv}}{\sum\limits_{j \in J} b_j} \tag{9}$$

$$Subject \, to: \; x_{jv} + \sum_{i \in I} a^v_{ij} y_i \geq 1 \; for \, j \in J, \, v = 0, 1, \dots, m - 1 \tag{10}$$

$$\sum_{i \in I} y_i = p \tag{11}$$

$$y_i \in \{0, 1\} \; for \, i \in I \tag{12}$$

$$x_{jv} \in \{0, 1\} \; for \, j \in J, \, v = 0, 1, \dots, m - 1 \tag{13}$$

The objective function (9) expresses the same average distance as the original objective function (1), but it does not need the allocation variables. The relationship between the variables x_{jv} and y_i is controlled via the linkage expressions (10). When at least one fire station is placed in the radius v from the served network node j, the variable x_{jv} can take a value of zero. Condition (11) maintains the rule that the highest number of located FPUs cannot be exceeded. The last two series of mandatory restrictive requirements (12) and (13) control the range of the variables y_i and x_{jv}.

If the distances are not integers or one is satisfied with an upper or a lower approximation of the distances, the above-introduced radial formulation can be generalized by using zones, as suggested and discussed in [30,31].

The presented location-allocation and radial models (1)–(6) and (9)–(13) suffer from one big weakness that significantly decreases their usability in the real world. It must be noted that the rescue system works as a queuing system. The requests for service occur randomly, and they can be neither expected nor directly estimated. Consequently, the nearest fire station may be temporarily unavailable at the time of a new emergency call, and thus, this station would not be able to accept the request for service. In such a situation, the demand is covered by the nearest available FPU, which does not have to be the closest-located one. The mentioned generalization is known as the concept of so-called generalized disutility, which has been studied in many papers, e.g., [30]. Let us recall its principle using a mathematical formulation.

Without any loss of generality, it is assumed that the *r* closest-located fire stations may participate in covering the current demand for service occurring anywhere. In the adjusted quality criterion, q_k stands for the likelihood that the *k*-th nearest FPU is the one that is closest, currently available, and may be able to meet increased service demand. This idea greatly increases the realism of the deterministic model because it considers the stochastic behavior of the real system at a certain level. The basic idea can be easily understood by looking at Figure 1, which was originally created for a medical rescue system.
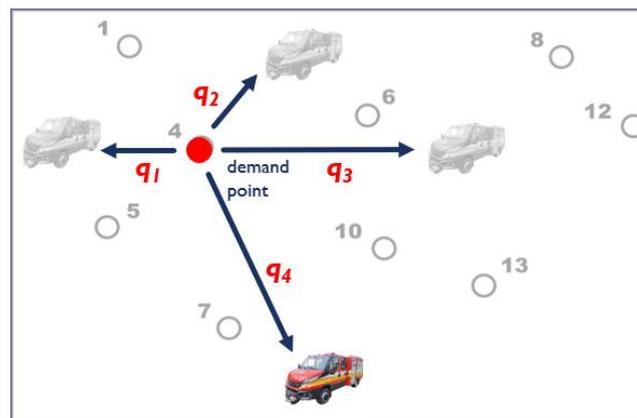

Figure 1. The main idea of generalized disutility.

The adjusted mathematical model, which takes into account the main idea of generalized disutility, takes the following form.

$$Minimize \quad \frac{\sum\limits_{j \in J} b_j \sum\limits_{v=0}^{m-1} \sum\limits_{k=1}^{r} q_k x_{jvk}}{\sum\limits_{j \in J} b_j} \tag{14}$$

$$Subject\ to: \quad \sum_{k=1}^{r} x_{jvk} + \sum_{i \in I} a_{ij}^v y_i \geq r\ for\ j \in J,\ v = 0, 1, \ldots, m-1 \tag{15}$$

$$\sum_{i \in I} y_i = p \tag{16}$$

$$y_i \in \{0, 1\}\ for\ i \in I \tag{17}$$

$$x_{jvk} \in \{0, 1\}\ for\ j \in J,\ v = 0, \ldots, m-1,\ k = 1, \ldots, r \tag{18}$$

The objective function (14) expresses the average distance from served network nodes to the nearest available FPU. The constraints (15) ensure that the sum of variables x_{jvk} over *k* expresses the number of fire stations in the radius *v* from the network node *j*, which

remains as the number *r*. The constraints (16), (17), and (19) keep the same meaning as in the above mathematical models.

3. Fair Modeling Approaches

This section addresses possible ways to master the requirements of system users for fair access to service. Without mentioning it directly, it is obvious that the optimization of an average distance may lead to an unfair system design. The fire stations are located via the model in such a way that they are placed near big network nodes with high weight coefficients b_j. Thus, the model does not reflect any form of fairness at all. There may be many clients whose distances to the nearest FPU exceed the average value many times. Those people would consider the resulting system design unacceptable [9,32,33].

The strongest way of mastering fairness consists of the minimization of the maximal distance from served network nodes to the nearest FPU. For simplicity, we report the basic model without the concept of generalized disutility.

The min–max strategy attempts to reduce the worst distance by solving network nodes that are too far from a nearby fire station. The accompanying solution process looks for optimal sites for the centers. The model may, of course, employ the same notations as those first introduced for the previous location-allocation model (1)–(6). The aforementioned definitions apply to the variables y_i and z_{ij}. In order to find exactly *p* locations in the candidate list *I*, it is necessary to minimize the maximum disutility experienced by the user in the worst possible position. In addition to the above terms, it is essential to introduce a non-negative variable *h*, which represents the minimized upper bound of the distances between the served network nodes and their nearest service providers. This problem's specific location-allocation model can be expressed as (19)–(26).

$$Minimize \quad h \tag{19}$$

$$Subject\,to: \sum_{i \in I} z_{ij} = 1 \; for\, j \in J \tag{20}$$

$$\sum_{i \in I} y_i = p \tag{21}$$

$$z_{ij} \leq y_i \; for\, i \in I,\, j \in J \tag{22}$$

$$\sum_{i \in I} d_{ij} z_{ij} \leq h \; for\, j \in J \tag{23}$$

$$y_i \in \{0,\, 1\} \; for\, i \in I \tag{24}$$

$$z_{ij} \in \{0,\, 1\} \; for\, i \in I,\, j \in J \tag{25}$$

$$h \geq 0 \tag{26}$$

As the model (19)–(26) is based on the model (1)–(6), no additional explanation is required. The upper bound of all perceived disutility values is provided by the objective function (19), which is represented as a single variable, *h*. Each perceived disutility is kept below or equal to the upper bound *h* thanks to the constraints (23). The mandatory constraint (26) is superfluous because the value of *h* is contained in (23).

Two key issues need to be addressed practically. The first of them arises from the need for allocation variables z_{ij} as in model (1)–(6). Therefore, the solution to real cases usually requires unacceptable amounts of memory and computational time. The second inconvenience follows from the model structure, mainly from the link-up constraints (23). Simple reformulation into a radial form as before (see the models (1)–(6) and (9)–(13)) will not help now, because the complicated model structure remains unchanged (see constraints (23) and the objective function (19)). Binding structural constraints may help to explain the branch and bound method's sluggish convergence. Looking closely at the model's structure, we discover that the radius formulation of the original model enables a straightforward simplification of the problem without the requirement for the upper bound

h and constraints (23). The transformation's core is in utilizing the beneficial aspects of covering issues and is predicated on the following: The values of the variables x_{jv} in the radial model form a specific sequence with a certain structure. Sorting out the variables x_{jv} according to the index v from 0, the corresponding list of items then has the form $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$. Therefore, in the beginning, the variables have a value of one, and, occasionally, they change their values and keep being zero [31]. This observation is then the key to transforming the former min–max model into a covering model.

The minimization of the biggest distance can be realized via the following iterative approach: From the sequence of distances $d_0, d_1, \ldots, d_m$, where $0 = d_0$ and $m = d_m$, we choose the "middle" division point D_s. For this distance, we solve a simple coverage model (27)–(31), based on which we will see whether such a feasible solution to the problem in which all distances between served network nodes and their service providers (fire stations) do not exceed the value of D_s exists or not. If there is such a solution, all variables x_j will take a value of zero. In the opposite case, at least one variable x_j that is not zero will occur. We determine the coverage coefficients a_{ij} for the given D_s based on a simple property: if $d_{ij} \leq D_s$, then $a_{ij} = 1$; otherwise, $a_{ij} = 0$. The problem solved for the dividing point D_s can then be expressed as (27)–(31).

$$Minimize \quad \sum_{j \in J} x_j \tag{27}$$

$$Subject\ to: \quad x_j + \sum_{i \in I} a_{ij} y_i \geq 1 \quad for\ j \in J \tag{28}$$

$$\sum_{i \in I} y_i = p \tag{29}$$

$$y_i \in \{0, 1\} \quad for\ i \in I \tag{30}$$

$$x_j \geq 0 \quad for\ j \in J \tag{31}$$

If in solving the model (27)–(31), we learn that the sum of variables x_j is zero, then we repeat the calculation for a lower element D_s; otherwise, we continue with a higher dividing point D_s. In this way, by combining the model (27)–(31) with the bisection method applied to the whole list of values $d_0, d_1, \ldots, d_m$, we can easily obtain the minimum distance of the worst-situated nodes from their closest fire stations. Solving the min–max problem by repeatedly computing a simple coverage problem takes a considerably shorter time than solving the original model, even if formulated in a radial way.

We decided to report the min–max model as an extreme way to achieve a certain level of fairness. This approach is not suitable in practice for one simple reason. While the common weighted *p*-median model locates fire stations that are near big network nodes, the min–max approach tries to establish FPUs in such a way they would be near the worst-situated users. None of these concepts are acceptable for a certain group of people. Via the minimization of the highest distance, we worsen the average. Therefore, we present herein a compromise, which consists of extending the original model (1)–(6) or (9)–(13) with additional constraints. This way, the fairness demands will be expressed in the following way, which was originally suggested to optimize the Emergency Medical Service systems in Slovak self-governing regions [34].

Let us introduce a new parameter D (which does not necessarily need to be an integer), which represents a "critical" distance (see Figure 2). In addition, we set a parameter B to limit the number of customers or their locations that can be farther than D from any fire station.
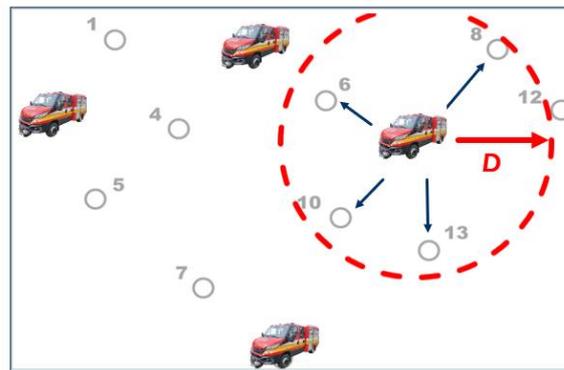
Figure 2. Visualization of fairness approach parameters.

A collection of binary variables z_j must be provided in order to represent the aforementioned fairness criterion using mathematical expressions. The variable $z_j \in \{0, 1\}$ is defined for each location of the clients $j \in J$. It takes a value of zero if the client's location j is covered by service within the radius D from the nearest fire station. As can be seen in Figure 2, the auxiliary variable z_{12} takes a value of one, and the variables z_6, z_8, z_{10}, and z_{13} take a value of zero. The mathematical expression can take the form of (32).

$$z_j + \sum_{\substack{i \in I \\ d_{ij} \leq D}} y_i \geq 1 \quad \text{for } j \in J \tag{32}$$

The number of system users whose distance to the nearest stations exceeds D can be limited via expression (33). It should be noted that the parameter B is given as a percentage, i.e., it takes the real value from the interval $[0, 1]$.

$$\sum_{j \in J} z_j b_j \leq B \sum_{j \in J} b_j \tag{33}$$

If one wanted to apply fairness demands less strongly for individual clients, but only for clients' locations regardless of the number of clients sharing the locations, this could be carried out by replacing the constraint (33) with the simpler expression (34).

$$\sum_{j \in J} z_j \leq B|J| \tag{34}$$

To complete the model with the mentioned extensive constraints, it is necessary to also add the obligatory constraints (35).

$$z_j \in \{0, 1\} \; for \, j \in J \tag{35}$$

When new fairness constraints are added to the preceding model (9)–(13), the original set of workable options is reduced. As a result, the expanded model's optimal solution might have a larger objective function value, which would worsen the average level of service accessibility for customers.

Inappropriate settings of parameters D and B may cause infeasibility of the problem because the given number p of fire protection units to be located may not be enough to cover the whole area in the required fair way. In such a case, it is useful to know the minimum required number of facilities to be able to meet the problem constraints.

The resulting objective function value of the following simple set covering model (36)–(40) can be used to determine the minimum number p_{min} of stations necessary to satisfy the fairness requirements. The set covering strategy is used in the model, and besides the

variables that were already included in the previous model, no more input data are needed.

$$Minimize \ \ p_{min} = \sum_{i \in I} y_i \tag{36}$$

$$Subject \ to: \ \ z_j + \sum_{\substack{i \in I \\ d_{ij} \leq D}} y_i \geq 1 \quad for \ j \in J \tag{37}$$

$$\sum_{j \in J} z_j b_j \leq B \sum_{j \in J} b_j \tag{38}$$

$$z_j \in \{0, 1\} \ for \ j \in J \tag{39}$$

$$y_i \in \{0, 1\} \ for \ i \in I \tag{40}$$

The minimized objective function (36) expresses the number of located fire stations. The structural constraints (37) and (38) are taken from the fairness expansion statement. Of course, expression (34) can replace constraint (38). The mandatory constraints (39) and (40) maintain the range of the decision variables used.

Since the problem described in terms (36)–(40) minimizes only the number of fire station locations and does not consider the average distance of customers to the nearest service providers, the model (9)–(13) with additional fairness constraints can be solved for p_{min} in order to obtain the optimal locations of the fire stations.

4. Small Case Study with Basic Models

Theoretical approaches to the search for optimal FPU deployment were explained in previous chapters. In this section, a small computational study is reported. Its main goal is to demonstrate how optimization may bring several changes and improvements to an existing system and how different parameters affect the resulting system design.

The preliminary numerical experiments were performed using the FICO Xpress 7.7 (64-bit, release 2014) optimization kit. Particular models were solved with a common PC with the 11th gen Intel® Core™ i7 1165G7 2.8 GHz CPU and 40 GB RAM.

The computational study begins with a brief analysis of current FPU deployment in a selected region. Even if we used the datasets of all regions depicted in Figure 3, the present tables would only report the results for the Žilina autonomous region. The other outcomes were quite comparable. Thus, we decided against publishing large tables with many similar numbers.

Figure 3. Self-governing regions of Slovakia.

The Žilina autonomous region is located in the northwestern part of the Slovak Republic (see Figure 3). The region includes five subareas (Horné Považie, Kysuce, Liptov, Orava, and Turiec), which are divided into eleven districts. In an area of 6809 km^2 (14% of the Slovak territory), 690,778 people live in the region, and the population density is 101.4 inhabitants/km^2. The counties of the Žilina autonomous region are shown in Figure 4.

Figure 4. Districts of the autonomous region of Žilina.

The region of Žilina is formed by 315 network nodes, which correspond to individual cities, villages, and small communities. For simplicity, we assume that each network node offers at least one possible location for establishing a fire station. Otherwise, the set of candidates would be necessary to redefine. Obviously, all inhabited nodes represent possible demand points in which an emergency may occur. Thus, sets I and J are the same. The weight coefficients b_j are set according to the number of inhabitants, and they are rounded up to hundreds since more precise data for fire brigade deployment (degree of fire risk, number of emergency calls, etc.) are currently not available. Let us now focus on the fire brigade system in Žilina.

The fire brigade system in the self-governing region of Žilina currently operates 15 stations, which are deployed in Bytča, including Čadca, Dolný Kubín, Kysucké Nové Mesto, Liptovský Hrádok, Liptovský Mikuláš, Martin, Námestovo, Rajec, Ružomberok., Terchová, Turčianske Teplice, Turzovka, Tvrdošín, and Žilina. The average distance between served network nodes and their nearest located fire station is 5.37, but the maximal relevant distance from a user's location to the nearest FPU is already 33. This fact confirms that there is at least one dwelling place or a small community that is too far from any service provider. As we can see, the longest distance exceeds the average value multiple times. Just for interest, the number of residents whose distance exceeds the value of 10 is 1407 (this number is given in 100s), which represents more than 20 percent of the served population. Obviously, such a simple analysis confirms the necessity of system optimization. Therefore, we performed some numerical experiments with the basic models.

The first experiment to report was aimed at solving the basic weighted p-median model (1)–(6) for $p = 15$ to verify the existence of a better solution, which could improve current fire brigade deployment. Since the solved problem was small, the computational time necessary for finding the optimal solution did not exceed 3.5 s. When we focused on the objective function (1), we could see that the average distance decreased from the original value of 5.37 to 5.31. This means that currently, the fire protection units are not deployed in an optimal way. The distance for the worst-situated users remained unchanged, and it still equaled 33. The percentage of clients beyond the 10 min limit was 19.39. Finally, we compared the resulting system designs, i.e., the list of network nodes with a fire station located. The optimization process of the location-allocation model (1)–(6) produced a slightly different result. The original centers in Námestovo, Terchová, and Liptovský Hrádok are suggested to be replaced with those in Krušetnica, Zubrohlava, and Belá. To sum up the first numerical experiment, the exchange of three fire stations may bring an improvement in service accessibility and decrease the number of users whose distance exceeds ten.

Even though the average distance as the studied quality criterion had acceptably small values in both the current and the optimal system designs, and thus the obtained solutions were satisfactory, we concentrated on studying how the number of located stations impacts the studied objective function value. Therefore, we performed a series of computations with different values of p starting from five. The obtained results are summarized in

Table 1, which has the following structure. Each row of the table corresponds to one problem instance with a different value of parameter p. For each problem instance, the computational time in seconds is reported in the column denoted by *CT*. The objective function value is given in the next column denoted by *AvgDist*. Analogically, we use *DmM* to denote the maximal relevant distances, i.e., the distances from the worst-situated users to their nearest FPUs. Finally, the last column denoted by *PB* contains the percentage of users behind the limit of 10 min from the nearest fire station.

Table 1. Results of numerical experiments with the location-allocation model (1)–(6) using the dataset for the self-governing region of Žilina.

p	CT	AvgDist	DmM	PB
5	3.48	12.31	45	50.57
6	3.17	10.7	42	44.97
7	3.19	9.24	42	40.96
8	3.34	8.49	42	37.17
9	3.11	7.76	42	33.56
10	3.19	7.28	38	30.70
11	3.31	6.81	33	28.90
12	3.70	6.37	33	26.97
13	3.00	5.99	33	24.42
14	3.20	5.65	33	22.10
15	3.80	5.31	33	19.39
16	2.95	4.98	33	17.00
17	2.99	4.79	33	16.13
18	2.96	4.61	33	15.74
19	2.96	4.44	26	14.46
20	2.98	4.28	26	13.30

The data reported in Table 1 show expected trends. The higher the number of located fire stations, the shorter the average distance because the fire brigades are near potential demand points. Despite this positive message, the highest distance remains too big and much above the average.

The second reported portion of experiments was aimed at the minimization of the maximal distance, i.e., for the model (19)–(26) with possible considerable acceleration with the series of models (27)–(31). Those models are suggested to find a system design in which the maximal relevant distance cannot be decreased with a given number of stations. As before, we solved the models for different values of parameter p. To save space, we report Table 2 in a different format. The objective function of the model is reported herein in rows denoted by *DmM*. For each resulting system design, the average distance was computed, and its value is given in the row denoted by *AvgDist*. Thus, each column of the table corresponds to one solved problem instance.

Table 2. Results of numerical experiments with *minMax* model (19)–(26) using the dataset for the self-governing region of Žilina.

p	5	6	7	8	9	10	11	12	13	14	15
DmM	41	34	34	31	28	27	25	23	23	23	22
AvgDist	17.50	15.21	13.75	12.38	11.53	11.40	11.67	12.04	9.59	9.32	10.67

The achieved results are interesting from an analytical point of view. Even if the maximal relevant distance is the smallest possible, the average distance values do not seem very good. This can be explained by the fact that the model minimizes only the highest distance, and it does not care about the average. Thus, applying one criterion may lead to worsening the average value of distances. If we compare the results in Tables 1 and 2, we can see that the minimization process of the highest relevant distance using the model (19)–(26) caused a significant increase in the average distances. For practical usage, it would be useful to combine these two approaches and create a two-stage method [35]. In the first step, the maximal distance could be minimized, and then the average value could be optimized under the condition of not worsening the previously found limit for the maximal distance. Another possibility consists of incorporating some form of fairness constraints into the basic model, as suggested at the end of the previous chapter.

5. Advanced Challenges and Future Modeling Directions

The previous parts of this paper were focused on applications of the basic modeling concepts in the field of fire protection units. Obviously, all suggested mathematical models were based on a significant level of abstraction. This means that many important aspects of the real system could not be taken into account.

Strategic decisions, which are the core output of the optimization process, are usually made for long periods (several years), and thus, the methodology of searching for the optimal structure of fire stations plays an important role. Let us suggest some new views on station network optimization.

5.1. System Robustness

Certain standard circumstances are typically anticipated for a network via which the service is offered while creating or redesigning a fire system. This indicates that no unforeseen issues or service delivery delays are assumed. However, this assumption is not correct and does not apply to all types of service systems. It must be understood that the emergency service system runs continuously for 24 h a day, 365 days a year. Logically, many different undesirable effects that can occur randomly anywhere and at any time can adversely affect the performance of the system. Such critical adverse events represent uncertainty, the source of which can be divided into two main groups:

1. **Endogenous (internal) causes**: These adverse events are due to the service system itself. At the time the fire protection unit is requested, staff may be temporarily unavailable because they are serving another user of the system. To include such situations in the decision-making process, the concept of so-called *generalized disutility* was introduced.

2. **Exogenous (external) causes**: This group of problems includes various impairments of transportation networks due to the time of day (morning and afternoon rush hour), weather, or technical condition of roads. Such events are independent of the designed system but may negatively affect its performance. Separate distance or time matrices usually describe such critical events and thus form so-called harmful scenarios. The goal of the solution process is, then, to find a mode of service center deployment that minimizes the mentioned negative effects of the harmful scenarios.

The time perspective is an additional factor that needs to be taken into account. The solution reached has strategic relevance since changes in service center deployment brought about by the results of the related decision process are frequently anticipated to continue for a very long time. System robustness should therefore be taken into account. The goal of system robustness is to make the designed system as resistant to unpredictable and destructive events as humanly possible. Let us discuss a quick, real-world example.

Simple examples of various factors that can negatively impact the operational efficiency of a service system and cause high traffic volumes in the corresponding transportation network are shown in Figure 5.

Figure 5. Uncertainty following randomly occurring failures in the network.

As was already indicated, when making decisions during the strategic stage of system design, it is important to take detrimental occurrences that happen at random into account. The following diagram illustrates how the ideal placement of two service centers is impacted by the extension of a network arc. The little numbers at the network nodes signify how many service requests are there in each node. The standard conditions and optimal distribution for the two service centers are shown in the left portion of Figure 6 after the weighted two-median problem is solved. The identical network graph is shown on Figure 6's right side but with a longer arc brought on by a negative event. As can be observed, the optimal solution to the problem is dramatically altered by the proposed arc evaluation, and the resulting system designs are different.
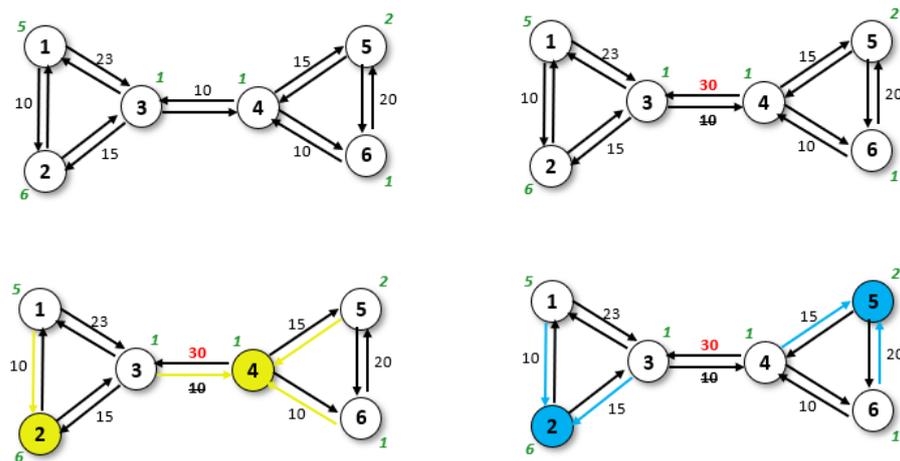
Figure 6. Impact of a detrimental event on the optimal centers' deployments.

In the practical application of robust service systems, there are usually more disjoint harmful events and much larger networks. Therefore, finding the optimal use of service centers in terms of the robustness of the systems is a challenging task for experts in the field of operations research. Another big challenge consists of generating scenarios, which could follow an advanced analysis of the associated transportation network.

To incorporate system resilience into the associated mathematical model, a finite set of damage scenarios must be formed that covers the most commonly expected disruptions to service delivery. In this way, the associated public service system can be mitigated by minimizing the highest adverse impact of each scenario.

5.2. System Reengineering

The basic idea of system reengineering follows from the analysis of current service stations' deployments, which may not be optimal due to changing demands (changed weight coefficients of the served nodes) and the development of an underlying transportation network. Another reasonable explanation could be based on the assumption that any kind of optimization may bring a system design that differs a lot from the current deployment. Too many changes require a lot of financial costs, and they may be hardly accepted by the public even if the new system design could be much better.

To illustrate the problem of system reengineering in more detail, consider the simple example in Figure 7. Suppose the diagram on the left represents a current zoom out of the four fire stations highlighted in blue. Every vertex represents a possible demand point or network node. The sum of the distances from each network node to the nearest fire station was used as a quality metric to evaluate the current deployment. A value of 66 is assumed here. If we allow the current position of the firefighting unit to change and move the station from node 2 to node 6, we can redesign the system and achieve better values for the considered criteria. The new system design is shown in the diagram on the right and has a score of 64. This small example demonstrates the principles and goals of system reengineering. It is not a big change, but the effect is noticeable.
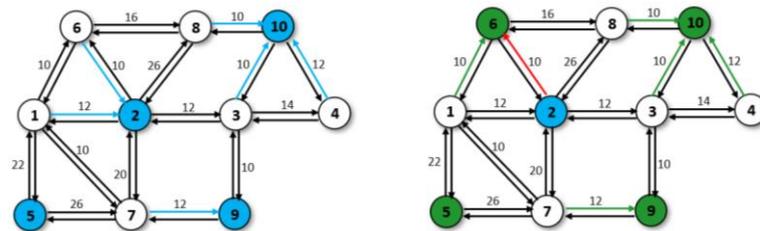
Figure 7. Simple example of system reengineering.

The basic idea of system reengineering can be achieved by solving the following radial model, which follows the original approach described in (14)–(18). Herein, a small adjustment must be performed. The original set *I* of candidates for locating a fire station needs to be divided into two separate disjoint sets, I_R and I_F. While the set I_F contains locations in which fire protection units must be located (or must stay unchanged from the current deployment), the set I_R contains nodes that are necessary to decide on. Based on these preliminaries, the radial model for system reengineering with generalized disutility may take the form of (41)–(45).

$$Minimize \quad \frac{\sum\limits_{j \in J} b_j \sum\limits_{v=0}^{m-1} \sum\limits_{k=1}^{r} q_k x_{jvk}}{\sum\limits_{j \in J} b_j} \tag{41}$$

$$Subject \, to : \sum_{k=1}^{r} x_{jsk} + \sum_{i \in I_R} a_{ij}^s y_i + \sum_{i \in I_F} a_{ij}^s \geq r \; for \, j \in J, \, v = 0, 1, \ldots, m-1 \tag{42}$$

$$\sum_{i \in I} y_i = p \tag{43}$$

$$y_i \in \{0, 1\} \; for \, i \in I \tag{44}$$

$$x_{jvk} \in \{0, 1\} \; for \, j \in J, \, v = 0, \ldots, m-1, \, k = 1, \ldots, r \tag{45}$$

Basic system reengineering may be accompanied by auxiliary constraints, which raise some extra restrictions. Let us mention two of them. The first rule limits the total number *w* of fire stations, the locations of which can change. The second rule limits the distance between the current and newly suggested location of a fire brigade. In addition to the sets I_F and I_R, let us define the set I_L as those currently deployed fire stations that are allowed to change their locations.

To formulate the rules concisely, we derive several auxiliary structures using Figure 8. We assume that all points 1–11 represent served network nodes and the black points 2, 3, 9, and 11 represent current FPU locations.
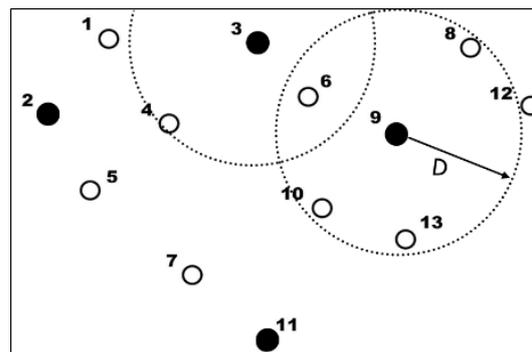
Figure 8. Simple example of reengineering restrictions. The black filled nodes represent currently deployed centers and the other nodes are used as candidate locations.

Let $N_t = \{i \in I_R: d_{ti} \leq D\}$ denote the set of all possible locations to which a fire station $t \in I_L$ can be moved subject to the limited length of the move. If we consider the example depicted in Figure 8, we can observe that the center located at point 9 can be moved to 6, 8, 10, and 13 or stay unchanged. Thus, the set $N_9 = \{6, 8, 9, 10, 13\}$. Similarly, $N_3 = \{3, 4, 6\}$. Additionally, let $S_i = \{t \in I_L: i \in N_t\}$ denote a set of all fire stations, which can be moved to $i \in I_R$ subject to the mentioned limitation. Here, $S_6 = \{3, 9\}$. Realize that $t \in N_t$ and $i \in S_i$ for $t \in I_L$ and $i \in I_R$, and thus $I_L \subset I_R$.

The choice to move centers from their current positions to new ones is now modeled with a set of decision reallocation variables. If the fire station at t is moved to i, the variable $u_{ti} \in \{0, 1\}$ for $t \in I_L$ and $i \in N_t$ takes a value of 1. Using the above-introduced structures and variables, we suggest the following model extension.

$$\sum_{i \in I_L} y_i \geq p - w \tag{46}$$

$$\sum_{i \in N_t} u_{ti} = 1 \; for \; t \in I_L \tag{47}$$

$$\sum_{t \in S_i} u_{ti} \leq y_i \; for \; i \in I_R \tag{48}$$

$$u_{ti} \in \{0, \; 1\} \; for \; t \in I_L, \; i \in N_t \tag{49}$$

Expression (46) prohibits having more than w modified FPU locations. An FPU can be moved from its current location t to a maximum of one other potential site within radius D thanks to constraints (47). As long as the moved station's initial location is within radius D, constraints (48) allow for the transportation of a maximum of one center to place i. In addition, these restrictions guarantee consistency between decisions about center location and motion.

5.3. Multi-Objective Optimization

The fact that only one goal is followed at a time is one of the main shortcomings of practically all of the modeling and solution approaches discussed in this study. Fire departments and many other services are far more complex; thus, it would be too strict to reduce them to a single goal. Hence, creating modeling and solution strategies that enable us to achieve appropriate outcomes for issues where at least two separate criteria must be taken into account at once will be a future research task for professionals in operations research, mathematics, and applied computer science. Let us think about the following brief illustration.

Whenever the decision-making task of the best possible deployment of fire protection units is formulated in a general way, it can take the form of (50).

$$\min\left\{ f(\mathbf{y}) : y_i \in \{0,\ 1\}, i \in I, \sum_{i \in I} y_i = p \right\} \tag{50}$$

The objective function $f(\mathbf{y})$ used in model (1) may take several different forms, and it directly affects the solvability of the problem using the available approaches. If we wanted to make the system of fire brigades' design problem more general, we could add at least one extra objective. The former problem (50) can be rewritten in the form of (51).

$$\min\left\{ f_1(\mathbf{y}), f_2(\mathbf{y}) : y_i \in \{0,\ 1\}, i \in I, \sum_{i \in I} y_i = p \right\} \tag{51}$$

The combination of two criteria may bring several difficulties into the decision-making process, mainly in situations in which the used objectives contradict each other. This means that improving one of them is not possible without worsening the other one.

Let us concentrate on the set of all feasible solutions to the problem (51) with any form of used objective functions. The vector $\mathbf{y}$ of the location variables y_i for $i \in I$ can describe each feasible solution. Since each solution $\mathbf{y}$ can be evaluated with two objectives, $f_1(\mathbf{y})$ and $f_2(\mathbf{y})$, each element of the feasible solutions set can be visually reported with one point of two-dimensional Euclidean space. The large cardinality of the input set I causes the impossibility of processing the complete set of feasible solutions. Thus, a suitable output of any multi-criteria optimization consists of a small subset called Pareto front [22–26].

A Pareto front is generally formed by a small number of solutions satisfying the *non-dominance* property for each pair of its members. It should be noted that in bi-criteria optimization, each feasible solution P can be evaluated with two criteria, $f_1(P)$ and $f_2(P)$, regardless of their form. A solution P can be considered a non-dominated solution if every other solution R for which $[f_1(P), f_2(P)] \neq [f_1(R), f_2(R)]$ satisfies the following inequality $f_1(P) < f_1(R)$ or $f_2(P) < f_2(R)$. The explanation of the Pareto front can be easily understood by looking at Figure 9.
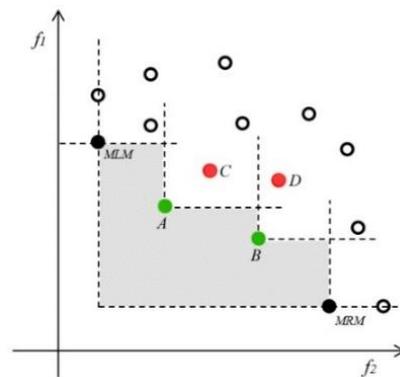
Figure 9. Pareto front of non-dominated solutions.

If we look at Figure 9, we can see that the green solutions A and B are members of the Pareto front because they are not dominated by any other solution. This means that none of the Pareto front members are equally good or better in both quality criteria. On the contrary, the red solutions C and D do not belong to the Pareto front. Particle A dominates solution C, and solution D is dominated by both particles A and B. For completeness, let us note that *MLM* and *MRM* represent the bordering members of the Pareto front. While *MLM* denotes the most left member, the symbol *MRM* is used to denote the most right one. The bordering members can be obtained by optimizing only one of the given criteria.

One of the next research challenges could consist of creating new exact and heuristic approaches for bi- or even multi-objective optimization problems to fulfill current trends in location science. One particular goal could be aimed at combining optimization with simulation verification as in [36], or we could focus on artificial intelligence and self-learning algorithms as suggested in [37–39].

5.4. Other Optimization Challenges

Fire brigade deployment represents a complex system, the optimization of which deserves proper planning, analysis, and a suitable level of abstraction. The existing system of fire stations could be analyzed from many other points of view than those described in this paper. The time accessibility of service is not the only possible criterion. As mentioned in the introductory part of this paper, individual fire stations are not equally equipped with technical devices and employees as well. Thus, the type of station should be taken into account. Another optimization approach may be based on an iterative approach. This means that the stations should be located first, and then, personal and technological resources may be distributed to those stations. As an advanced strategy, we could distinguish between professional and volunteer fire brigades. Since the real system is very complex, it may be useful to identify its critical features that should not be abstracted while the optimization process is being performed. An irreplaceable role could be played in the proper consideration of network nodes' degree of fire risk.

To sum up the proposed modeling and optimization challenges, it is natural that the operations research field has a lot to offer. If everything turns out well, remarkable results can be achieved. Several applicable research directions also in the field of searching for optimal fire brigade deployment have been proposed in recent publications [40–44].

6. Conclusions

The optimization of fire brigade' deployment plays an important role from many points of view. This research paper was aimed at two levels of mathematical modeling. The first part of this paper summarized the basic modeling concepts, which were accompanied by a small computational study. We showed concrete ways in which simple models can deploy fire brigades. The second half of this paper introduced several advanced challenges, which could bring novelty into the design of rescue service systems.

After the introductory section, explained several basic modeling concepts were explained. The mathematical models were discussed in detail together with possible modifications and reformulations. The second part was focused on theoretical advanced challenges. We discussed system robustness and the concept of so-called reengineering, which follows from the existing system and limits the number of performed changes in current fire brigade deployment. Finally, we presented the possibility of managing the problem of searching for a new mode of fire brigade deployment using a bi-criteria model. Since each modeling concept may produce very different results, a simulation model could be helpful to identify the best modeling strategy. It could also assist us in identifying some weak points of the suggested models to create a suitable and complex optimization method for firefighters.

Since many defined modeling concepts and optimization approaches need additional input data or network analyses, they are not reported within this computational study. For example, evaluating system robustness requires defining the set of detrimental scenarios. They can either result from a network analysis, or they can be created by experts in transportation science. Due to lack of data, we would like to analyze system robustness in a separate article, in which we will discuss all problems that are connected to robust system designing. On the other hand, making emergency systems robust is an important concept; therefore, we decided to also report this in this paper, at least theoretically.

Future research in this field could be aimed at the verification of several advanced modeling strategies by performing numerical experiments with real data and identifying the best strategy for practical usage. Another topic could focus on some possible adjustments of existing methods to produce results that are more precise or on developing other

algorithms that could work better or faster. Thus, the optimization of fire brigades is interesting for experts in more scientific fields.

Institutional Review Board Statement: The study was conducted according to the guidelines of the Declaration of Helsinki and approved by the Ethics Committee of the University of Žilina (protocol code 1/3/2023) on 1 March 2023.

Informed Consent Statement: Presented results do not contain studies with human or animal subjects. Informed Consent Statements are not relevant for this study.

Data Availability Statement: The datasets used and/or analyzed during the reported case study are available from the corresponding author on reasonable request.

Conflicts of Interest: All authors of the manuscript declare that they have no conflict to report. We honestly declare that the presented results do not contain studies with human or animal subjects.

References

1. Brotcorne, L.; Laporte, G.; Semet, F. Ambulance location and relocation models. *Eur. J. Oper. Res.* **2003**, *147*, 451–463. [CrossRef]
2. Current, J.; Daskin, M.; Schilling, D. Discrete network location models. In *Facility Location: Applications and Theory*; Drezner, Z., Hamacher, H.W., Eds.; Springer: Berlin/Heidelberg, Germany, 2002; pp. 81–118.
3. Drezner, T.; Drezner, Z. The gravity p-median model. *Eur. J. Oper. Res.* **2007**, *179*, 1239–1251. [CrossRef]
4. Elloumi, S.; Labbé, M.; Pochet, Y. A new formulation and resolution method for the p-center problem. *INFORMS J. Comput.* **2004**, *16*, 84–94. [CrossRef]
5. Chanta, S.; Mayorga, M.E.; McLay, L.A. Improving emergency service in rural areas: A bi-objective covering location model for EMS systems. *Ann. Oper. Res.* **2011**, *221*, 133–159. [CrossRef]
6. Ingolfsson, A.; Budge, S.; Erkut, E. Optimal ambulance location with random delays and travel times. *Health Care Manag. Sci.* **2008**, *11*, 262–274. [CrossRef] [PubMed]
7. Reuter-Oppermann, M.; Berg, P.V.D.; Vile, J.L. Logistics for Emergency Medical Service systems. *Health Syst.* **2017**, *6*, 187–208. [CrossRef]
8. Snyder, L.V.; Daskin, M.S. Reliability models for facility location: The expected failure cost case. *Transp. Sci.* **2005**, *39*, 400–416. [CrossRef]
9. Janáček, J.; Buzna, Ľ. *Facility Location in Distribution Systems*; EDIS: Žilina, Slovakia, 2007; 142p, ISBN 978-80-8070-649-4.
10. Jankovič, P. Calculating Reduction Coefficients for Optimization of Emergency Service System Using Microscopic Simulation Model. In Proceedings of the 17th International Symposium on Computational Intelligence and Informatics, Budapest, Hungary, 17–19 November 2016; pp. 163–167.
11. Janosikova, L. Emergency Medical Service Planning. *Commun.—Sci. Lett. Univ. Zilina* **2007**, *9*, 64–68. [CrossRef]
12. Jánošíková, Ľ.; Žarnay, M. Location of emergency stations as the capacitated p-median problem. *Quant. Methods Econ. (Mult. Criteria Decis. Mak. XVII)* **2014**, *28*, 117–123.
13. Matiaško, K.; Kvet, M. Medical data management. In Proceedings of the 2017 IEEE International Scientific Conference on Informatics, Poprad, Slovakia, 14–16 November 2017; Institute of Electrical and Electronics Engineers: Danvers, MA, USA, 2017; pp. 253–257.
14. Doerner, K.F.; Gutjahr, W.J.; Hartl, R.F.; Karall, M.; Reimann, M. Heuristic Solution of an Extended Double-Coverage Ambulance Location Problem for Austria. *Cent. Eur. J. Oper. Res.* **2005**, *13*, 325–340.

15. Marianov, V.; Serra, D. Location problems in the public sector. In *Facility Location: Applications and Theory*; Drezner, Z., Hamacher, H.W., Eds.; Springer: Berlin/Heidelberg, Germany, 2002; pp. 119–150.

16. Schneeberger, K.; Doerner, K.; Kurz, A.; Schilde, M. Ambulance location and relocation models in a crisis. *Cent. Eur. J. Oper. Res.* **2014**, *24*, 1–27. [CrossRef]

17. Karatas, M.; Yakıcıa, E. An analysis of p-median location problem: Effects of backup service level and demand assignment policy. *Eur. J. Oper. Res.* **2019**, *272*, 207–218. [CrossRef]

18. Jánošíková, Ľ.; Kvet, M.; Jankovič, P.; Gábrišová, L. An optimization and simulation approach to emergency stations relocation. *Cent. Eur. J. Oper. Res.* **2019**, *27*, 737–758. [CrossRef]

19. Jánošíková, Ľ.; Jankovič, P.; Kvet, M.; Zajacová, F. Coverage versus response time objectives in ambulance location. *Int. J. Health Geogr.* **2021**, *20*, 32. [CrossRef] [PubMed]

20. Kvet, M.; Janáček, J. Reengineering of the Emergency Service System under Generalized Disutility. In Proceedings of the ICORES 2018: 7th International Conference on Operations Research and Enterprise Systems, Funchal, Portugal, 24–26 January 2018; pp. 85–93.

21. Kvet, M.; Janáček, J.; Kvet, M. Computational Study of Emergency Service System Reengineering Under Generalized Disutility. In Proceedings of the Operations Research and Enterprise Systems: 7th International Conference, ICORES 2018 Funchal, Madeira, Portugal, 24–26 January 2018; Revised Selected Papers; Springer: Berlin/Heidelberg, Germany, 2019; pp. 198–219, ISBN 978-3-030-16034-0.

22. Arroyo, J.E.C.; dos Santos, P.M.; Soares, M.S.; Santos, A.G. A Multi-Objective Genetic Algorithm with Path Relinking for the p-Median Problem. In Proceedings of the 12th Ibero-American Conference on Advances in Artificial Intelligence, Bahía Blanca, Argentina, 1–5 November 2010; pp. 70–79.

23. Grygar, D.; Fabricius, R. An efficient adjustment of genetic algorithm for Pareto front determination. In *TRANSCOM 2019: Conference Proceedings*; Elsevier Science: Amsterdam, The Netherlands, 2019; pp. 1335–1342.

24. Janacek, J.; Fabricius, R. Public service system design with conflicting criteria. *IEEE Access* **2021**, *9*, 130665–130679. [CrossRef]

25. Janáček, J.; Kvet, M. Emergency Medical System under Conflicting Criteria. In Proceedings of the SOR 2021 Proceedings, Online, 22–24 September 2021; pp. 629–635.

26. Kvet, M.; Janáček, J. Incrementing Heuristic for Non-Dominated Designs of Emergency Medical System. In Proceedings of the SOR 2021 Proceedings, Online, 22–24 September 2021; pp. 429–474.

27. Avella, P.; Sassano, A.; Vasil'Ev, I. Computational study of large-scale p-Median problems. *Math. Program.* **2006**, *109*, 89–114. [CrossRef]

28. García, S.; Labbé, M.; Marín, A. Solving large *p*-median problems with a radius formulation. *INFORMS J. Comput.* **2011**, *23*, 546–556. [CrossRef]

29. Janáček, J. Approximate Covering Models of Location Problems. In *Lecture Notes in Management Science, Proceedings of the 1st International Conference ICAOR, Yerevan, Armenia, 15–17 September 2008*; Tadbir Institute for Operational Research, Systems Design, and Financial Services: Vancouver, BC, Canada, 2008; pp. 53–61.

30. Kvet, M. Computational Study of Radial Approach to Public Service System Design with Generalized Utility. In Proceedings of the International Conference DT 2014, Žilina, Slovakia, 9–11 July 2014; pp. 198–208.

31. Kvet, M. Advanced radial approach to resource location problems. In *Developments and Advances in Intelligent Systems and Applications*; Studies in Computational Intelligence; Springer International Publishing: Cham, Switzerland, 2018; Volume 718, pp. 29–48.

32. Bertsimas, D.; Farias, V.F.; Trichakis, N. The Price of Fairness. *Oper. Res.* **2011**, *59*, 17–31. [CrossRef]

33. Buzna, Ľ.; Koháni, M.; Janáček, J. Proportionally Fairer Public Service Systems Design. *Commun.—Sci. Lett. Univ. Žilina* **2013**, *15*, 14–18. [CrossRef]

34. Kvet, M. Impact of Fairness Constraints on Average Service Accessibility in Emergency Medical System. In Proceedings of the 2021 International Conference on Information and Digital Technologies (IDT), Zilina, Slovakia, 22–24 June 2021; pp. 11–18.

35. Kvet, M.; Janáček, J. Composed Min-Max and Min-Sum Radial Approach to the Emergency System Design. In *Operations Research Proceedings 2015*; Springer: Cham, Switzerland, 2016; pp. 41–47.

36. Jankovič, P.; Jánošíková, Ľ. Ambulance Locations in a Tiered Emergency Medical System in a City. *Appl. Sci.* **2021**, *11*, 12160. [CrossRef]

37. Jalil, H.; Li, K.; Ying, W. Theoretical Analysis and Empirical Validation of the Conical Area Evolutionary Algorithm for Bi-Objective Optimization. *IEEE Trans. Evol. Comput.* **2022**, *26*, 1058–1071. [CrossRef]

38. Khaleghzadeh, H.; Fahad, M.; Shahid, A.; Manumachu, R.R.; Lastovetsky, A. Bi-Objective Optimization of Data-Parallel Applications on Heterogeneous HPC Platforms for Performance and Energy through Workload Distribution. *IEEE Trans. Parallel Distrib. Syst.* **2020**, *32*, 543–560. [CrossRef]

39. Yang, L.; Sun, Q.; Zhang, N.; Li, Y. Indirect Multi-Energy Transactions of Energy Internet with Deep Reinforcement Learning Approach. *IEEE Trans. Power Syst.* **2022**, *37*, 4067–4077. [CrossRef]

40. Altay, G.; Akyüz, M.H.; Öncan, T. Solving a minisum single facility location problem in three regions with different norms. *Ann. Oper. Res.* **2023**, *321*, 1–37. [CrossRef]

41. Blanco, V.; Gázquez, R.; Saldanha-Da-Gama, F. Multi-type maximal covering location problems: Hybridizing discrete and continuous problems. *Eur. J. Oper. Res.* **2023**, *307*, 1040–1054. [CrossRef]

42. Kvet, M. Overview of basic criteria and models applicable to emergency medical service system optimization. *SN Comput. Sci.* **2023**, *4*, 75. [CrossRef]
43. Kvet, M.; Janáček, J. Directed Search for Non-Dominated Emergency Medical System Designs. *Appl. Sci.* **2023**, *13*, 4810. [CrossRef]
44. Gonda, D.; Majer, T. Implementation of learning algorithm for rescue system with uncertainty. In Proceedings of the International Carpathian Control Conference 2023, Szilvásvárad, Hungary, 12–14 June 2023. in print.