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An Integral Method for Natural Convection of Van Der Waals Gases over a Vertical Plate

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Abstract: This paper focuses on a study of natural convection in a van der Waals gas over a vertical heated plate. In this paper, for the first time, an approximate analytical solution of the problem was obtained using an integral method for momentum and energy equations. A novel simplified form of the van der Waals equation for real gases enabled estimating the effects of the dimensionless van der Waals parameters on the normalized heat transfer coefficients and Nusselt numbers in an analytical form. Trends in the variation of the Nusselt number depending on the nature of the interaction between gas molecules and the wall were analyzed. The results of computations for a van der Waals gas were compared with the results for an ideal gas.

Keywords: natural convection; van der Waals gas; analytical solution

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1. Introduction

Nowadays, the need for knowledge of the physical and chemical properties of gases, the features of their behavior in real production conditions, and during their transportation and storage has significantly increased. The rational and efficient design of various technological processes involving gases can significantly reduce the cost of industrial production.

In technical thermodynamics, it is customary to refer to real gases as the gaseous state of any substance in the entire range of its existence, that is, at any pressures and temperatures. Under appropriate conditions, real gas can be liquefied or converted into a solid state. It is expected that the development of small-scale energy in the coming years will be associated with the widespread use of liquefied natural gas, which is recognized as one of the most promising types of energy carriers. The liquefaction of gases is important when storing and transporting this type of fuel. In its liquefied form, natural gas occupies only about 1/600 of its gaseous volume; therefore, it is easier and more economical to transport it [1–3].

The vapors of various substances, such as water, ammonia, methyl chloride, sulfur dioxide, and others are widely used in technological applications. The most widely used is water vapor, which is the main working medium of steam engines, heating, and other devices [4–7].

It is known that the ideal gas model allows a satisfactory description of the state of real gases only in a relatively small region of the variation of the state parameters. To design and determine the optimal operating conditions for heat exchange equipment, calculations must take into account the real thermophysical properties of the gas. The use of the ideal gas equation can lead to a significant error in determining the parameters of the gas state, up to 100%. The authors in [8] emphasized that for simple, mono-, and diatomic gases, such as air, helium, nitrogen, etc., under certain conditions, the model for ideal gases is well-suited. For gases with a more complex organic structure, the effects of real gases are more significant.

Taking air as an example, as a mixture of mono- and diatomic gases, the limits of the applicability of the ideal gas model for air are low densities (large specific volumes ν), low pressures (<10 bar), and moderate/high temperatures (up to 600 °C) [9,10].

From the point of view of the molecular theory of the structure of matter, a real gas is a gas, the properties of which depend on the interaction and size of the molecules. To date, more than 150 equations of state for real gases are known. One of the most widely known approximate equations of state for real gases is the van der Waals equation [9,10]. Some of the other equations of state for real gases are refinements of the van der Waals equation, while others were obtained experimentally.

Three forms of the van der Waals equation are used by different authors [9–13]:

$$\left(p + n^2 \frac{a_*}{V^2}\right)(V - nb_*) = nR_m T, \tag{1}$$

$$\left(p + \frac{a_*}{V_m^2}\right)(V_m - b_*) = R_m T \tag{2}$$

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT, \tag{3}$$

where $V_m = V/m = M_i/\rho$ is molar volume, $n = m/M_i$ is the number of moles, whereas $R = R_m/M_i$ and $R_m = 8314$ J/(mol·K).

Because the first and the second derivatives of p with respect to ν at the critical point must be zero, one can obtain the following relations, which link the empirical constants a and b (or a_* and b_*) in the Equations (1)–(3) with the parameters in the critical point [11–13]

$$p_R = \frac{a_*}{27b_*^2},\ T_R = \frac{8a_*}{27R_m b_*}, \tag{4}$$

$$a = 3p_{cr}v_{cr}^2,\ b = \frac{v_{cr}}{3},\ Z_{cr} = \frac{p_{cr}v_{cr}}{RT_{cr}} = \frac{3}{8}, \tag{5}$$

$$a = \frac{a_*}{M_i^2},\ b = \frac{b_*}{M_i}. \tag{6}$$

Here the compressibility factor Z accounts for the deviation of a real gas from ideal-gas behavior at a given temperature and pressure.

$$Z = \frac{p}{RT\rho} = \frac{pv}{RT\rho} = \frac{pV}{mRT}. \tag{7}$$

The experimentally measured values of the compressibility factor Z_{cr} for real gases vary over the range $Z_{cr} = 0.2 \ldots\ 0.3$. To make the van der Waals equation of state more accurate for each particular real gas, the constants a and b are determined via comparisons with precise experiments over a wider range of the parameters instead of from a single point [11–13].

For ideal gases, $Z = 1$ by definition, which results in the known ideal-gas equation [9,10]:

$$pV = m\frac{R_m}{M_i}T,\ pV = nR_m T,\ pv = \frac{R_m}{M_i}T = RT. \tag{8}$$

where $n = m/M_i$ is the number of moles.

For example, in recent works [14–16] on battery design, the van der Waals equation was used as a real gas model versus an ideal gas model. In these studies, various trends in the behavior of a real gas during its compression and expansion are presented and analyzed in both isothermal and adiabatic processes.

In the studies [16,17], several equations of state of a real gas are used, taking into account the interaction of water vapor molecules. The actual physical properties of humid air have been determined, and their influence on the conjugate heat and mass transfer for various conditions has been assessed. This provided an increase in the accuracy of predictions of heat and mass transfer processes when designing contact heat exchangers, convective drying plants, hygroscopic desalination plants, compressors with the injection of water or steam, as well as combustion chambers where flue gases are mixed with steam.

Heat transfer during natural and mixed convection has been studied extensively. In particular, some works [18–24] investigated the cases of natural and mixed convection in different geometries for the different ranges of physical parameters in the frames of the ideal gas model. The cases of real gases were considered in the studies [25–27].

Avramenko et al. [28] analytically solved the problem of natural convection in van der Waals gases near a heated vertical plate. Based on the use of the simplified van der Waals equation, analytical solutions were obtained for the profiles of velocity, temperature, and normalized Nusselt numbers. The limits of the applicability of the simplified van der Waals equation were determined. The data obtained were compared with the ideal gas model.

The objective of this work is to further study the influence of the thermophysical properties of a real gas in the framework of the above-mentioned simplified van der Waals model on heat transfer during natural convection near a heated vertical plate. A novel solution to the problem will be obtained for the first time using the integral method and compared with the previously obtained analytical solution. To authors' knowledge, such a solution has not been published in the literature yet.

2. Mathematical Model

We will here solve a problem of the steady-state natural convection over a vertical heated plate with the temperature T_w located in a non-moving gas, whose temperature is also the constant T_∞. We assume that $T_w > T_\infty$, however, the solution and its results obtained below will be valid for the case of $T_\infty > T_w$ as well. As a result of heating, a boundary layer of heated gas with a thickness of δ is formed near the plate, is formed near the plate with a vertical lifting movement affecting it. In the coordinate system we have chosen, the origin is at the lower edge of the plate, the x-axis is directed longitudinally upward, and the y-axis is directed perpendicular to the plate (Figure 1). The problem is solved in a two-dimensional statement under the assumption that the plate is infinite in the z-direction.

In the present paper, we investigate the influence of the thermophysical properties of a gas within the framework of the van der Waals equation of state on the characteristics of natural convection in comparison with the case of an ideal gas. Therefore, as a basis for comparison, we take the results for an ideal gas, which are also obtained based on the integral approach.

Within the framework of the adopted model, we assume that the physical properties of the gas, except for the density, are constant. In this regard, we only consider the buoyancy arising from the dependence of density on temperature. The energy dissipation is not considered, since the flow rate with free convection is small. The considered process of free convection is stationary.

As a result, the temperature and velocity fields can be described by the following differential equations in the boundary layer approximation [29]:

$$\frac{\partial u}{\partial x} + \frac{\partial \mathrm{v}}{\partial y} = 0, \tag{9}$$

$$u\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(1 - \frac{\rho}{\rho_\infty}\right), \tag{10}$$

$$u\frac{\partial T}{\partial x} + \mathrm{v}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}, \tag{11}$$

where x and y are the Cartesian coordinates, u and v are the streamwise (along the x-coordinate) and normal (along the y-coordinate) velocity components, respectively, and T is the local temperature, ν is the kinematic viscosity, g is the gravitational acceleration, ρ is the density, α is the thermal diffusivity, and the subscript "∞" refers to the parameters outside of the boundary layer.

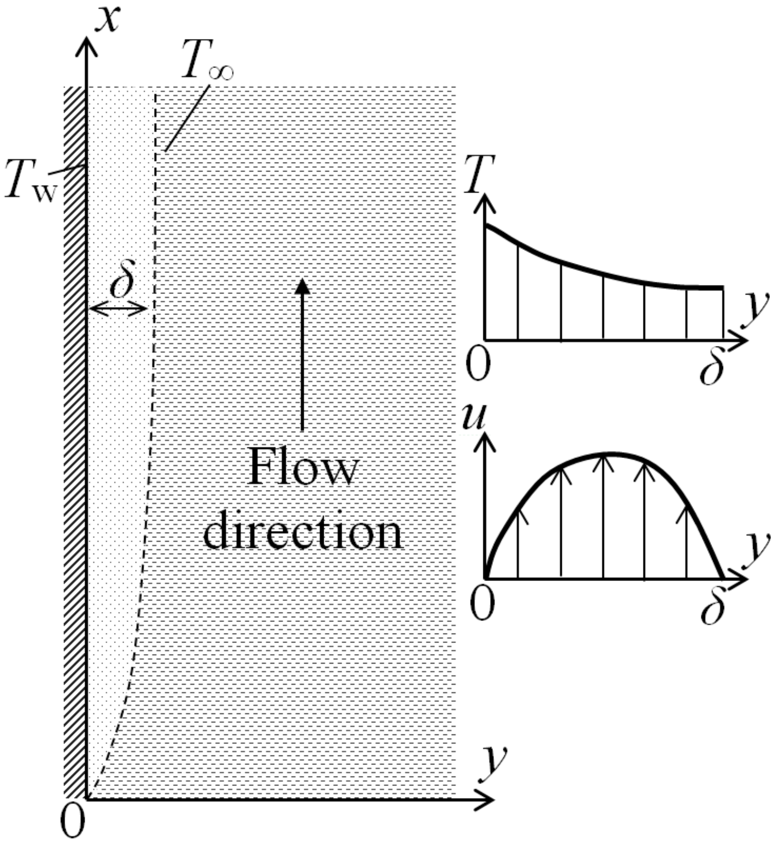

Figure 1. Schematic representation of two-dimensional natural convection boundary layer near a heated vertical plate.

As one can see from Equations (9) to (11), this model considers inertial terms and convective heat transfer. The lifting force is taken into account based on the last term of Equation (10).

In contrast to the simplified problem statement in the existing work [28], the problem statement in the present study considers the complete system of the equations of continuity, momentum, and energy, in which the inertial and convective components are explicitly taken into consideration.

The boundary conditions for system (9)–(11) are set as follows:

$$u = 0,\ T = T_w,\ \text{at}\ y = 0, \tag{12}$$

$$u = 0,\ T = T_\infty,\ \text{at}\ y = \delta, \tag{13}$$

where δ is the boundary layer thickness, and the subscript "w" refers to the parameters at the wall. The boundary conditions for the x-coordinate are described below in Section 3.

This problem statement must be completed with the van der Waals equation of state (3), rewritten using the gas density ρ instead of the specific volume υ:

$$\left(p + a\rho^2\right)\left(\frac{1}{\rho} - b\right) = RT, \tag{14}$$

where p is pressure, R is the individual (specific) gas constant, and a and b are the van der Waals constants.

The last term on the right-hand side of the equation of motion (10) can be transformed using the van der Waals equation of state (14). To do this, we solve Equation (14) with respect to density. The result is a cubic equation that has two complex conjugate roots and one real root.

$$\rho = \frac{1}{6b}\left(\begin{array}{c} 2 + \dfrac{\sqrt[3]{16}(a-3b(bp+RT))}{\sqrt[3]{2a^3+9a^2b(2bp-RT)-\sqrt{a^3\left(a(2a+9b(2bp-RT))^2-4(a-3b(bp+RT))^3\right)}}} + \\ + \dfrac{\sqrt[3]{2a^3+9a^2b(2bp-RT)-\sqrt{a^3\left(a(2a+9b(2bp-RT))^2-4(a-3b(bp+RT))^3\right)}}}{a\sqrt{8}} \end{array} \right), \tag{15}$$

The solution to this equation is rather cumbersome, so it is difficult to use it when integrating the system of Equations (9)–(11). Avramenko et al. [28] showed that in the approximation of small values of the constants a and b, Equation (15) can be expanded in a Maclaurin series, which, in what follows, considers only the first three terms:

$$\rho = \frac{p}{RT} - \frac{bp^2}{R^2T^2} + \frac{ap^2}{R^3T^3}, \tag{16}$$

This equation can be represented in the dimensionless form [28]:

$$Z(1 + \mathrm{Wa}_a - \mathrm{Wa}_b) = 1, \tag{17}$$

where

$$\mathrm{Wa}_a = \frac{ap}{R^2T^2}, \quad \mathrm{Wa}_b = \frac{bp}{RT}, \tag{18}$$

are van der Waals numbers.

Avramenko et al. [28] validated the simplified van der Waals Equation (16) for ethylene. It was shown that Equation (16) is more accurate than the ideal gas Equation (8), and agrees well with the full van der Waals Equation (8) up to the pressure of p = 40 bar (or p/p_{cr} = 0.4) and T = T_{cr}. In this case, the relative error of Equation (16) is about 8%, whereas the relative error of the ideal gas Equation (8) is 22.7%.

As it was demonstrated by Avramenko et al. [28], for these pressures and temperatures, the values of both van der Waals numbers for ethylene are $\mathrm{Wa}_a = 2.78 \cdot 10^{-6}$ and $\mathrm{Wa}_b = 9.03 \cdot 10^{-7}$. This completely justifies the assumption of the small values of the parameters Wa_a and Wa_b that lie in the background of the simplified van der Waals Equation (16) in the series form.

The density ratio ρ/ρ_∞ in Equation (10) can be expressed using Equations (16)–(18). As a result, we get:

$$u\frac{\partial u}{\partial x} + \mathrm{v}\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(1 - \frac{Z}{1+\beta\Delta T\theta}\left(1 - \frac{\mathrm{Wa}_b}{1+\beta\Delta T\theta} + \frac{\mathrm{Wa}_a}{(1+\beta\Delta T\theta)^2}\right)\right), \tag{19}$$

Here, the compressibility factor Z and van der Waals numbers Wa_a and Wa_b are defined using the parameters outside of the boundary layer (subscript "∞").

The dimensionless local temperature in the boundary layer is defined as:

$$\theta = \frac{T - T_\infty}{T_w - T_\infty} = \frac{T - T_\infty}{\Delta T}, \quad \Delta T = T_w - T_\infty, \tag{20}$$

whereas

$$\beta = \frac{1}{T_\infty}, \tag{21}$$

is the volume expansion coefficient [9–11].

3. Integral Forms of Equations

The solution of the stated problem will be performed using the integral method. The essence of the integral method is the reduction in the original system of differential transport equations in partial derivatives into a system of ordinary differential equations, in which the marching coordinate x remains the only independent variable [29,30]. To do this, one should assume proper approximating functions for the profiles of the longitudinal velocity component u and the local temperature T in the boundary layer, and integrate transport Equations (10) and (11) within the limits from the wall to the outer boundary δ of the boundary layer. In this case, the continuity equation is incorporated into the integral equation of momentum transfer (to exclude the transverse velocity component v). The unknown quantities in the so-called integral equations of the boundary layer (which, in fact, after integration across the boundary layer, become ordinary differential equations) are: (a) the boundary layer thickness δ, which for gases is the same for the velocity and temperature profiles; and (b) the scaling (maximum) velocity in the boundary layer U, which increases with the development of the boundary layer and is itself unknown, and which must be found from the solution of the problem. Therefore, the only boundary conditions along the marching coordinate x, which are needed for solving the integral equations of the boundary layer, are the equality to zero of the thickness of the boundary layer δ and the scaling velocity U at the leading edge of the plate at $x = 0$. These boundary conditions will be implicitly incorporated into the mathematical form of the functions $\delta(x)$ and $U(x)$ in the subsequent solution of the problem.

Integral equations for the boundary layer in the case of the free convection of an ideal gas are given in the classical works [30,31]. Let us derive integral equations for the case of a real gas considered in this paper and described by the van der Waals equation. To obtain the integral equation of motion, we multiply the continuity Equation (9) by the velocity component and add it to the left side of Equation (19). As a result, we have:

$$\frac{\partial u^2}{\partial x} + \frac{\partial u\mathrm{v}}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(1 - \frac{Z}{1+\beta\Delta T\theta}\left(1 - \frac{\mathrm{Wa}_b}{1+\beta\Delta T\theta} + \frac{\mathrm{Wa}_a}{(1+\beta\Delta T\theta)^2}\right)\right), \tag{22}$$

Let us integrate Equation (22) over the thickness of the boundary layer, taking into account the boundary conditions (12) and (13). This gives:

$$\frac{d}{dx}\int_0^\delta u^2 dy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + g\int_0^\delta \left(1 - \frac{Z}{1+\beta\Delta T\theta}\left(1 - \frac{\mathrm{Wa}_b}{1+\beta\Delta T\theta} + \frac{\mathrm{Wa}_a}{(1+\beta\Delta T\theta)^2}\right)\right) dy, \tag{23}$$

Let us further rewrite this equation using dimensionless variables and functions:

$$\frac{d\left(\delta U^2\right)}{dx}\int_0^1 w^2 d\eta = -\nu\left(\frac{\partial w}{\partial \eta}\right)_{\eta=0}\frac{U}{\delta} + g\delta\int_0^1 \left(1 - \frac{Z}{1+\beta\Delta T\theta}\left(1 - \frac{\mathrm{Wa}_b}{1+\beta\Delta T\theta} + \frac{\mathrm{Wa}_a}{(1+\beta\Delta T\theta)^2}\right)\right) d\eta, \tag{24}$$

where U is a scaling velocity that depends on the streamwise coordinate (see below), whereas:

$$w = \frac{u}{U}, \qquad \eta = \frac{y}{\delta}, \tag{25}$$

In a similar way, one can integrate the energy Equation (11). This brings:

$$\frac{d(U\delta)}{dx}\int_0^1 w\theta d\eta = -\frac{\alpha}{\delta}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0}, \tag{26}$$

To solve Equations (24) and (26), it is necessary to specify the velocity and temperature profiles in the boundary layer. Authors [30,31] proposed the following equations for these profiles:

$$w = \eta(1-\eta)^2, \tag{27}$$

$$\theta = (1-\eta)^3, \tag{28}$$

The substitution of profiles (27) and (28) into the integral Equation (24) gives:

$$\frac{1}{105}\frac{d\left(\delta U^2\right)}{dx} = -\nu\frac{U}{\delta} + g\delta F, \tag{29}$$

where

$$F(\beta\Delta T,\mathrm{Wa}_a,\mathrm{Wa}_b)=\frac{1}{54}\left(54-\frac{3(-6\mathrm{Wa}_b(1+\beta\Delta T)+\mathrm{Wa}_a(8+5\beta\Delta T))}{(1+\mathrm{Wa}_a-\mathrm{Wa}_b)(1+\beta\Delta T)^2}+\frac{(9+5\mathrm{Wa}-6\mathrm{Wa}_b)\left(6\arctan\left(\frac{1-2\sqrt[3]{\beta\Delta T}}{\sqrt{3}}\right)+\sqrt{3}\ln\left(1-\frac{3\sqrt[3]{\beta\Delta T}}{\left(1+\sqrt[3]{\beta\Delta T}\right)^2}\right)-\pi\right)}{\sqrt[3]{\beta\Delta T\sqrt{3}}(1+\mathrm{Wa}_a-\mathrm{Wa}_b)}\right) \tag{30}$$

In the limiting case of $\beta\Delta T\to 0$, Equation (29) transforms into:

$$\frac{1}{105}\frac{d\left(\delta U^2\right)}{dx}=-\nu\frac{U}{\delta}+\frac{1}{4}g\delta\beta\Delta T\frac{1+3\mathrm{Wa}_a-2\mathrm{Wa}_b}{1+\mathrm{Wa}_a-\mathrm{Wa}_b}. \tag{31}$$

For an ideal gas (Wa_a = Wa_b = 0), Equation (31) reduces to the respective equation obtained in the classical works [30,31]. Energy Equation (26) takes the following form:

$$\frac{1}{42}\frac{d(\delta U)}{dx}=3\frac{\alpha}{\delta}, \tag{32}$$

If Equation (28) for the temperature profile is replaced by another profile used in our study [28]:

$$\theta=(1-\eta)^2, \tag{33}$$

then the form of the function F in Equation (30) will change as follows:

$$F(\beta\Delta T,\mathrm{Wa}_a,\mathrm{Wa}_b)=1-\frac{-4\mathrm{Wa}_b(1+\beta\Delta T)+\mathrm{Wa}_a(5+3\beta\Delta T)}{8(1+\mathrm{Wa}_a-\mathrm{Wa}_b)(1+\beta\Delta T)^2}-\frac{(8+3\mathrm{Wa}_a-4\mathrm{Wa}_b)\arctan\left(\sqrt{\beta\Delta T}\right)}{8\sqrt{\beta\Delta T}(1+\mathrm{Wa}_a-\mathrm{Wa}_b)} \tag{34}$$

Then, in the limiting case of $\beta\Delta T\to 0$ used in the Boussinesq approximation [30,31], the integral Equation (29) looks as follows:

$$\frac{1}{105}\frac{d\left(\delta U^2\right)}{dx}=-\nu\frac{U}{\delta}+\frac{1}{3}g\delta\beta\Delta T\frac{1+3\mathrm{Wa}_a-2\mathrm{Wa}_b}{1+\mathrm{Wa}_a-\mathrm{Wa}_b}, \tag{35}$$

and the energy equation takes the form:

$$\frac{1}{30}\frac{d(\delta U)}{dx}=2\frac{\alpha}{\delta}, \tag{36}$$

Summing up, we can write the system of integral equations in a general form:

$$s\frac{d\left(\delta U^2\right)}{dx}=-\nu\frac{U}{\delta}+g\delta F, \tag{37}$$

$$t\frac{d(\delta U)}{dx}=z\frac{\alpha}{\delta}, \tag{38}$$

where the function F is defined either by Equation (30) or Equation (34). Here s and t are numerical coefficients. Substituting Equation (27) into Equation (24), one can obtain s = 1/105; a substitution of Equation (27) into Equation (24) yields t = 1/30.

4. Heat Transfer

Following the authors [30,31], we will solve the system (37) and (38) using power-law equations for the scaling velocity U and the boundary layer thickness δ:

$$U(x)=\Psi x^m, \tag{39}$$

$$\delta(x)=\Phi x^n, \tag{40}$$

where m, n, Ψ, and Φ are yet unknown constants. Solutions (39) and (40) satisfy the requirement that the scaling velocity U and the boundary layer thickness δ are equal to zero at the leading edge of the plate at $x = 0$. To find these constants, we substitute Equations (39) and (40) into Equations (37) and (38). As a result, we have:

$$(2m+n)s\Phi\Psi^2 x^{2m+n-1} = -\nu\frac{\Psi^2}{\Phi}x^{m-n} + gF\Phi x^n, \tag{41}$$

$$(m+n)t\Phi\Psi x^{m+n-1} = \alpha\frac{z}{\Phi}x^{-n}. \tag{42}$$

Let us equate the exponents at the coordinate x. As a result, we have a system of equations, whose solution is:

$$m = \frac{1}{2}, \qquad n = \frac{1}{4}, \tag{43}$$

Substituting these values into Equations (41) and (42), we obtain a system of equations for the unknown constants Ψ and Φ. Eliminating one of these unknowns, we obtain a fourth-order algebraic equation with respect to the other unknown. As a result of the solution, we have four pairs of roots, of which only one has a positive real form:

$$\Psi = 2\sqrt{\frac{zgF}{5sz+3t\mathrm{Pr}}}, \tag{44}$$

$$\Phi = \sqrt{\frac{4z}{3t}\frac{\alpha}{\Psi}}, \tag{45}$$

Having obtained relations (44) and (45), it is possible to derive a solution for the heat transfer coefficient:

$$h = \frac{k}{\delta}\left(\frac{d\theta}{d\eta}\right)_{\eta=0} = z\frac{k}{\delta} = k\sqrt{\frac{3zt}{4}\frac{\Psi}{\alpha\sqrt{x}}} = k\sqrt{\frac{3zt}{2}\frac{1}{\alpha\sqrt{x}}\sqrt{\frac{zgF}{5sz+3t\mathrm{Pr}}}}, \tag{46}$$

Equation (46) can be rewritten in the form of the normalized Nusselt number:

$$\frac{\mathrm{Nu}}{\mathrm{Nu}_0} = \sqrt[4]{\frac{F}{r\beta\Delta T}}, \tag{47}$$

where

$$\mathrm{Nu} = \frac{hx}{k} \tag{48}$$

The Nusselt number is commonly interpreted as a dimensionless heat transfer coefficient, which characterizes the heat transfer rate at the boundary between the wall and the flow [29–31].

Here, the subscript "0" refers to the ideal gas ($\mathrm{Wa}_a = \mathrm{Wa}_b = 0$).

The coefficient r is determined from the equation:

$$\lim_{\beta\Delta T\to 0}\frac{F}{\beta\Delta T} = r\frac{1+3\mathrm{Wa}_a-2\mathrm{Wa}_b}{1+\mathrm{Wa}_a-\mathrm{Wa}_b}, \tag{49}$$

For example, $r = 1/4$ for Equation (30), whereas $r = 1/3$ for Equation (34).

As a result, Equation (47) can be rewritten as follows:

$$\frac{\mathrm{Nu}}{\mathrm{Nu}_0} = \sqrt[4]{\frac{1+3\mathrm{Wa}_a-2\mathrm{Wa}_b}{1+\mathrm{Wa}_a-\mathrm{Wa}_b}}, \tag{50}$$

where Nu and Nu_0 are the Nusselt numbers for the van der Waals gas and the ideal gas, respectively.

In the integral method considered here, the main sought quantity is the Nusselt number. If necessary, the final relation for the velocity profile can be easily found using the profile (27) for the dimensionless velocity w, as well as the combination of Equations (39) and (44) for the function U. The profile of the local temperature T is found from either Equation (28) or Equation (33), whereas the gas density ρ is determined by Equation (16).

5. Results and Discussion

Figures 2 and 3 elucidate the effect of the dimensionless van der Waals numbers on the normalized Nusselt number Nu/Nu_0, as predicted by Equation (50). The numerical values of Nu/Nu_0 coincide with the data obtained in our work [28] using an approximate analytical solution.

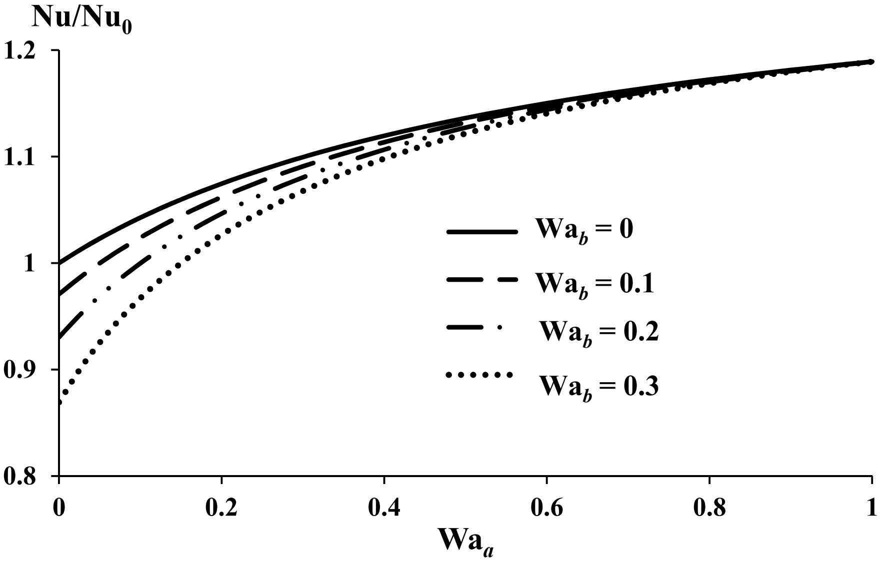

Figure 2. Effect of the van der Waals number Wa_a on the normalized Nusselt number under condition Wa_b = const: (1) $Wa_b = 0$; (2) $Wa_b = 0.1$; (3) $Wa_b = 0.2$; (4) $Wa_b = 0.3$.

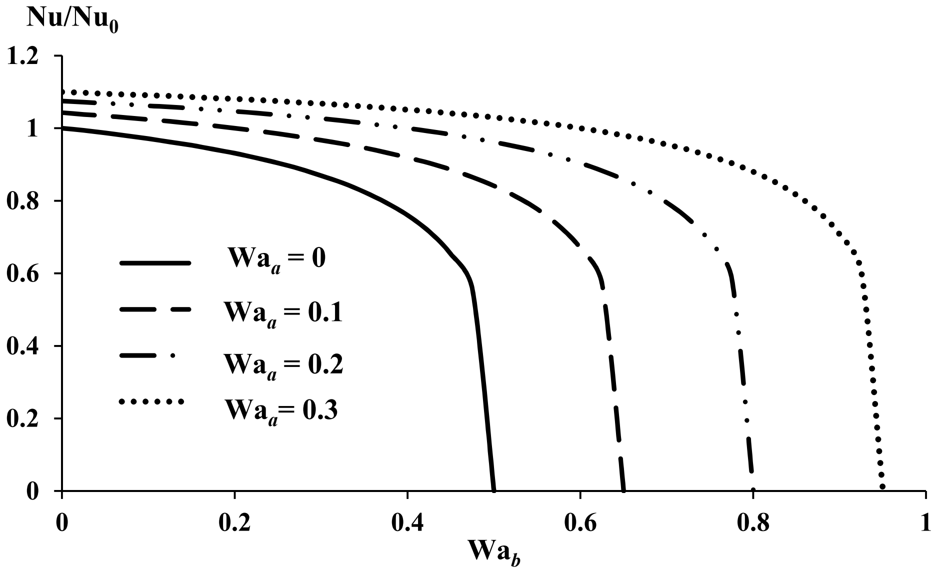

Figure 3. Effect of the van der Waals number Wa_b on the normalized Nusselt number under condition Wa_a = const: (1) $Wa_a = 0$; (2) $Wa_a = 0.1$; (3) $Wa_a = 0.2$; (4) $Wa_a = 0.3$.

The parameter a in the van der Waals equation characterizes the additional pressure in the near-wall layer, which increases the probability of collision of real gas molecules with the wall in comparison with the ideal one. Obviously, this leads to an increase in lifting force. As a result, the velocity in the boundary layer increases, which causes an increase in heat transfer with an increase in the Wa_a number.

As it is known, the nature of the interaction of gas molecules with the wall has a significant effect on heat transfer, which occurs due to the exchange of energy between molecules and the surface. The parameter a in the van der Waals equation characterizes the additional pressure in the near-wall layer, which increases the probability of a collision between real gas molecules and the wall in comparison with the ideal one. Obviously, this leads to an increase in the lifting force. As a result, the velocity in the boundary layer increases, which causes an increase in heat transfer, with an increase in the Wa_a number.

In turn, the parameter b describes the additional volume not filled with molecules. With its increase, the heat transfer between the molecules and the wall decreases, which causes deterioration in the conditions of interaction between them. Consequently, as can be seen from Figures 2 and 3, with an increase in the Wa_b number, the normalized Nusselt number decreases compared to ideal gas case.

Calculations have shown that with an increase in the Wa_a number, the effect of the Wa_b number noticeably weakens, and at values of $\mathrm{Wa}_a \geq 0.6$, cannot be observed at all (Figure 2). The more pronounced effect of the Wa_a number on the flow characteristics can be explained by the fact that the constant a describes the quadratic effect on the density variation in the van der Waals equation, whereas the parameter b describes the linear effect on the density variation.

We must, however, emphasize that Equation (16) is to be used for small values of the parameters Waa and Wab for the approximate series expansion of the full van der Waals equation to remain in force. Thus, for the larger values of Waa and Wab, Figures 2 and 3 demonstrate only qualitative trends.

6. Conclusions

The study has dealt with the problem of the steady-state natural convection of a van der Waals gas near a vertical heated plate. The problem was solved analytically based on an integral method. The novel solution was obtained for the first time using a novel, simplified form of the van der Waals equation of state proposed in our work [28]. To the authors' knowledge, such a solution has not yet been published in the literature.

The effects of the dimensionless Wa_a and Wa_b numbers on the normalized Nusselt number in the real gas compared to the ideal gas were estimated. The analysis of the calculation results showed that, with an increase in the Wa_a number (which characterizes the additional pressure in the real gas), the normalized Nusselt number increases. This is due to an increase in the lifting force and velocity in the boundary layer. The effect of additional volume, which manifests itself in an increase in the Wa_b number, causes a deterioration in the conditions for interaction between gas molecules and the wall. This is accompanied by a decrease in the Archimedes force and flow rate in the boundary layer, which leads to a weakening of heat transfer when compared with an ideal gas.

It was therefore shown that the change in the heat transfer intensity is due to effects that are considered by the van der Waals equation of state, but not by the ideal gas equation.

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Abbreviations

a, b	van der Waals constants;
g	gravitational acceleration;
h	heat transfer coefficient;
k	thermal conductivity;
p	pressure;
p_{cr}	critical pressure;
R	individual (specific) gas constant;
T	local temperature;
T_{cr}	critical temperature;
u	streamwise velocity component;
v	spanwise (wall-orthogonal) velocity component;
V	volume;
υ	specific volume;
x	streamwise coordinate;
y	spanwise (wall-orthogonal) coordinate
Greek symbols	
α	thermal diffusivity;
δ	boundary layer thickness;
ν	kinematic viscosity;
ρ	density
Dimensionless values	
Nu	Nusselt number;
Pr	Prandtl number;
U	dimensionless axial (streamwise) velocity;
Θ	dimensionless temperature;
Wa_a, Wa_b	van der Waals numbers
Subscripts:	
0	ideal gas;
w	value of a parameter at the wall;
∞	value at the outer boundary of the boundary layer.

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