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Novel Adaptive Extended State Observer for Dynamic Parameter Identification with Asymptotic Convergence

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Abstract: In this paper, a novel method of parameter identification of linear in parameter dynamic systems is presented. The proposed scheme employs an Extended State Observer to online estimate a state of the plant and momentary value of total disturbance present in the system. A notion is made that for properly redefined dynamics of the system, this estimate can be interpreted as a measure of modeling error caused by the parameter uncertainty. Under this notion, a disturbance estimate is used as a basis for classic gradient identification. A global convergence of both state and parameter estimates to their true values is proved using the Lyapunov approach under an assumption of a persistent excitation. Finally, results of simulation and experiments are presented to support the theoretical analysis. The experiments were conducted using a compliant manipulator joint and obtained results show the usefulness of the proposed method in drive control systems and robotics.

Keywords: parameter identification; extended state observer; persistent excitation; lyapunov stability

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1. Introduction

The problem of parameter identification of dynamic systems and its extension to adaptive control constitute one of the basic branches of control theory. Different algorithms capable of recovering true values of dynamic system parameters have been proposed in the literature [1–3], including variations of classic gradient-based identification [4,5] and model reference adaptive control [6–8], or recent $\mathcal{L}_1$ adaptive control methods [9,10]. Such a recovery can be accomplished by using a certain measure of modeling error computed from measurable outputs. For this purpose, for example, tracking errors and signal deviations determined on the basis of some predefined reference dynamics can be considered.

Among others, adaptive observers, which support simultaneous estimation of the system parameters and their state, can be distinguished as a separate category. Solutions have been proposed for linear time-invariant [11–13], linear time-varying systems [14–16], state-affine [17], or fully nonlinear systems [18–21].

The continuous scientific interest in methods of system identification and adaptive control has led to multiple significant results regarding the performance and stability of such algorithms. One of the most important results was the introduction of the persistency of excitation condition in [22] stating that the control inputs of the dynamic plant have to be sufficiently rich to enable a proper identification of its parameters. This notion was later actively investigated [23–26] and also inspired a more detailed study on the properties of adaptive systems with respect to exponential and uniform stability [27–29]. Recently, the first stability analyzes based on the Lyapunov approach have also been presented [30,31]. In this paper, the recent methodology is used to analyze the stability of the proposed identification/estimation algorithm.

Another approach to the problem of modeling uncertainties is represented by a paradigm of Active Disturbance Rejection Control (ADRC) [32–34]. Instead of an explicit parametric identification of the plant, the ADRC approach redefines the system in such a way that all uncertain dynamics are lumped into a single additional state variable, called a *total disturbance*, which together with the original state of the system constitutes an observable *extended state*. Momentary values of this extended state are then estimated by an extended-state observer (ESO), which allows one to use the estimate of the total disturbance in a control scheme to compensate for any unknown dynamics. The ADRC approach can therefore offer a good performance without requiring explicit knowledge of the dynamic parameters of the system.

As the appearance of the ADRC paradigm more than two decades ago, many results have been reported on the stability and convergence characteristics of ESO-based control structures. In particular, it has been proved that the estimation errors of ESO converge to some neighborhood of the origin under the assumption that the derivative of a total disturbance is bounded [35–38] and this neighborhood can be made arbitrarily small by increasing observer gains [39]. It has been shown that asymptotic convergence of the estimation errors can be achieved if the total disturbance is constant or its derivative vanishes with time [40,41]. These properties comprise significant drawbacks of the ESO-based control in real-life scenarios, as the value of the total disturbance is seldom constant and observer gains cannot be increased at will due to measurement noise and discrete-time domain implementation issues.

In this paper, a novel method of dynamic parameter identification is proposed that combines the merits of both classic parameter identification and ESO-based estimation aproaches. The new Parameter Identifying ESO (PIESO) is designed for a linear-in-parameter dynamic system to simultaneously estimate its state and parameters. To synthesize this algorithm, the plant model is first rewritten to incorporate the estimates of the parameters, while the modeling error caused by the parameter uncertainty is considered to be a part of the total disturbance. The ESO is then used to compute an estimate of this disturbance, which is regarded as a measure used for online adaptation of the system parameters. The obtained parameters are then fed back into the ESO itself. In the proposed method, while both the state and the parameters are estimated, the two estimation processes are strongly separated, and a fast estimation of the state is achieved regardless of the parameter convergence speed. By means of the Lyapunov analysis, it is shown that under the persistence of excitation condition, the performance of ESO is improved as a result of parameter adaptation and the asymptotic convergence of the estimates of both states and parameters of the system is guaranteed regardless of whether the regressor depends on the estimates themselves or not.

Despite the intuitiveness of the notion of connection between the ADRC approach and adaptive control schemes, there have been only a few significant attempts to propose controllers utilizing the merits of both approaches. Some results on this topic were presented in [42] for a specific case of adaptive control of DC motors, where switching functions were used for an adaptation law. In recent work, [43] a similar algorithm was applied to the control of electro-hydraulic servomechanisms. In both works, only the convergence to some boundary of the origin was proven. Alternative approaches were also presented [44,45] in which an extremum seeking optimization scheme was employed to tune the model parameters in order to minimize a predefined cost function. Here, only a discrete-time scenario was considered, with results verified only by an empirical evaluation with no analytical stability proof. Moreover, in all of the aforementioned works, the identification and adaptation schemes were based on the tracking error instead of employing information on the estimated disturbance. This changed in [46] where the adaptation was carried out using an estimate of total disturbance, but the structure of the ESO itself was not adaptively modified. Moreover, only the identification of an input gain was performed, which was also a case in recent [47]. In [48] all parameters of the system were identified, but the structure of the ESO was still constant, and, thus, the asymptotic convergence was not obtained. In [49,50] an adaptive ESO was proposed and the least squares algorithm was employed based on the estimate of disturbance; however, it did not lead to an asymptotic convergence of the estimation errors. Moreover, in recent [51] an adaptive scheme was presented that employed an estimate of the total disturbance to achieve a global convergence of all signals, nonetheless the algorithm was designed for the specific problem of harmonic disturbance frequency identification only. This scenario is also addressed in this paper.

The rest of the paper is organized as follows. In Section 2, the considered problem is defined and some useful results, mostly already presented in the literature, are recalled. Section 3 presents the proposed solution along with the stability analysis. In Section 4, some use cases are shown and the results of the simulations are presented. Section 5 contains experimental validation of the proposed scheme. Section 6 concludes the paper and highlights plans for future work.

2. Background and Problem Formulation

2.1. Nomenclature

Throughout this paper the following vector-matrix notation is employed: $\mathbf{I}_{\kappa\times\kappa}$ and $\mathbf{0}_{\kappa\times\kappa}$ stand for the identity and zero matrices of size $\kappa\times\kappa$ correspondingly,

$$\forall\kappa\geq 2\in\mathbb{N},\ \mathbf{A}_\kappa=\begin{bmatrix}\mathbf{0}_{\kappa-1\times 1} & \mathbf{I}_{\kappa-1\times\kappa-1}\\ 0 & \mathbf{0}_{1\times\kappa-1}\end{bmatrix}\in\mathbb{R}^{\kappa\times\kappa},\ \mathbf{b}_\kappa=\begin{bmatrix}\mathbf{0}_{\kappa-1\times 1}\\ 1\end{bmatrix}\in\mathbb{R}^{\kappa},$$
$$\mathbf{c}_\kappa=\begin{bmatrix}1\\ \mathbf{0}_{\kappa-1\times 1}\end{bmatrix}\in\mathbb{R}^{\kappa},\ \mathbf{d}_\kappa=\begin{bmatrix}\mathbf{0}_{\kappa-2\times 1}\\ 1\\ 0\end{bmatrix}\in\mathbb{R}^{\kappa}.$$

For any constant or bounded matrix or vector $\mathbf{H}$, unless explicitly defined otherwise, denote the constant $h_{\mathrm{M}}\in\mathbb{R}_+$ such that $\|\mathbf{H}\|\leq h_{\mathrm{M}}$. For any constant matrix $\mathbf{H}$ denote also $h_{\mathrm{m}}\in\mathbb{R}_+$ satisfying $h_{\mathrm{m}}\leq\lambda_{\min}(\mathbf{H})$ with $\lambda_{\min}(\mathbf{H})$ being the smallest eigenvalue of $\mathbf{H}$. For any variable x, its estimate is denoted by $\hat{x}$ and $\tilde{x}=x-\hat{x}$ stands for an estimation error. Denote also a scaling matrix $\mathbf{\Phi}_\kappa(\omega)=\mathrm{diag}(\omega^{\kappa-1},\omega^{\kappa-2},\ldots,1)\in\mathbb{R}^{\kappa\times\kappa}$ for any constants $\kappa\in\mathbb{N}$ and $\omega\in\mathbb{R}_+$. Finally, define matrix $\mathbf{\Lambda}_\kappa=\begin{bmatrix}\mathbf{I}_{\kappa\times\kappa} & \mathbf{0}_{\kappa\times 1}\end{bmatrix}\in\mathbb{R}^{\kappa\times\kappa+1}$. For any function $f(t,x(t))$ the notations $\frac{d}{dt}f(t,x(t))$ and $\dot{f}(t,x(t))$ are used interchangeably to denote the total derivative with respect to time.

2.2. Process Model and Essential Assumptions

Consider the following dynamic system

$$\dot{\mathbf{x}}=\mathbf{A}_n\mathbf{x}+\mathbf{b}_n(bu+\Delta),\tag{1}$$

where $\mathbf{x}=\begin{bmatrix}x_1\ldots x_n\end{bmatrix}^T\in\mathbb{R}^n$ is a state, $b\in\mathbb{R}$ is a known constant input gain, $u\in\mathbb{R}$ is a control signal, and $\Delta\in\mathbb{R}$ stands for an additive term which can depend on the state and external signals. In this paper, the following essential assumptions with respect to Δ are taken into account.

Assumption 1 (Linear parametrization)**.** *Assume that the term Δ can be represented by the following linear in parameters form*

$$\Delta=\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta},\tag{2}$$

where $\boldsymbol{\psi}(t,\mathbf{x})=\begin{bmatrix}\psi_1(t,\mathbf{x})\ldots\psi_k(t,\mathbf{x})\end{bmatrix}\in\mathbb{R}^{1\times k}$ is a regressor, which in the general case can be dependent on both time and state variables while $\boldsymbol{\theta}=\begin{bmatrix}\theta_1\ldots\theta_k\end{bmatrix}^T\in\mathbb{R}^k$ stands for constant parameters.

Assumption 2 (Boundedness of the signals)**.** *Let $\boldsymbol{\psi}(t,\mathbf{x})\in\mathcal{C}^1$ and be Lipschitz with respect to $\mathbf{x}$, that is*

$$\|\boldsymbol{\psi}(t,\mathbf{x}_a)-\boldsymbol{\psi}(t,\mathbf{x}_b)\|\leq\psi_{\mathrm{M}}\|\mathbf{x}_a-\mathbf{x}_b\|\tag{3}$$

for any $\mathbf{x}_a$ and $\mathbf{x}_b$. Let the system (1) evolve in such a way that

$$\max\left(\|\boldsymbol{\psi}(t,\mathbf{x})\|_\infty, \left\|\frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\right\|_\infty\right) \leq \psi_{\mathrm{M}} \tag{4}$$

for some constant $\psi_{\mathrm{M}} \in \mathbb{R}_+$.

Assumption 3 (Persistency of excitation)**.** *Let the system (6) evolve in such a way that the regressor $\boldsymbol{\psi}(t,\mathbf{x}(t))$ satisfies*

$$\int_t^{t+T} \boldsymbol{\psi}^T(\tau,\mathbf{x}(\tau))\boldsymbol{\psi}(\tau,\mathbf{x}(\tau))d\tau \geq \mu\mathbf{I} \tag{5}$$

for some constants $\mu, T \in \mathbb{R}_+$ and all $t \geq 0$.

Remark 1. *Assumption 2 refers to the ADRC methodology and in the considered case is used to guarantee the boundedness of $\boldsymbol{\psi}(t,\mathbf{x})$ for the nominal system (1) along with (2). However, it does not impose strict limitations on the value of $\boldsymbol{\psi}(t,\mathbf{x})$ evaluated at different (possibly estimated) trajectories of $\mathbf{x}$. Assumption 3 is commonly used in the identification and requires the evolution of $\boldsymbol{\psi}(t,\mathbf{x})$ to be rich enough to enable the convergence of parameter estimates to the nominal values.*

From now on, the parametrization (2) is employed in (1), namely the nominal control process becomes

$$\dot{\mathbf{x}} = \mathbf{A}_n\mathbf{x} + \mathbf{b}_n(bu + \boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta}), \tag{6}$$

see also Figure 1 where the system structure is shown.

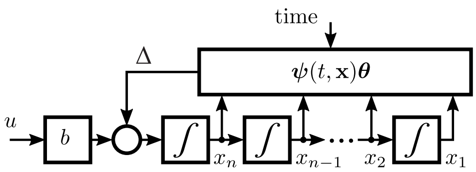

Figure 1. The dynamic system (1) along with the the assumed parametrization (2).

2.3. Extended State Observer for State Estimation

To facilitate the design of the ESO observer [52–54] consider a dynamic extension of system (6) under Assumption 2. In such a case, the extended state can be defined as $\mathbf{z} = \begin{bmatrix}\mathbf{x}^T & \delta\end{bmatrix}^T \in \mathbb{R}^m$, with $m = n+1$, and the extended dynamics of the plant can be rewritten as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_n\mathbf{x} + \mathbf{b}(bu + \delta), \\ \dot{\delta} &= \frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta} \end{aligned} \tag{7}$$

or, equivalently, using the following compact form

$$\dot{\mathbf{z}} = \mathbf{A}_m\mathbf{z} + \mathbf{d}_m bu + \mathbf{b}_m\frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta}, \tag{8}$$

where δ stands for the total disturbance in the system, which here directly corresponds to the additive term Δ. To estimate the extended state, an observer can be designed as

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}_m\hat{\mathbf{z}} + \mathbf{d}_m bu + \mathbf{l}(z_1 - \hat{z}_1), \tag{9}$$

where $\mathbf{l} = \begin{bmatrix} l_1 & \dots & l_m \end{bmatrix}^T \in \mathbb{R}_+^m$ are constant observer gains. Defining the estimation error as $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$, for the gains $\mathbf{l}$ chosen such that $\mathbf{A}_m - \mathbf{l}\mathbf{c}_m^T$ is Hurwitz, it can be proved that the time derivative of the function

$$V_{\text{eso}}(\tilde{\mathbf{z}}) = \frac{1}{2}\tilde{\mathbf{z}}^T\mathbf{P}\tilde{\mathbf{z}}, \tag{10}$$

satisfies

$$\dot{V}_{\text{eso}}(\tilde{\mathbf{z}}) = -\frac{1}{2}\tilde{\mathbf{z}}^T\mathbf{Q}\tilde{\mathbf{z}} + \tilde{\mathbf{z}}^T\mathbf{P}\mathbf{b}_m\frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta} \tag{11}$$

for some positive definite matrices $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{m\times m}$. From (10) and (11) it can be deduced that

$$\limsup_{t\to\infty}\|\tilde{\mathbf{z}}(t)\| = 2p_{\text{M}}q_{\text{m}}^{-1}\frac{d}{dt}\boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta} \tag{12}$$

and thus, the estimates of all state variables and the total disturbance converge to some neighborhood of their true values. Moreover, it can be shown that this boundary can be made arbitrarily small by increasing the gains $\mathbf{l}$. Although the required quality of the estimation could be obtained under the assumption that observer gains are selected high enough, this approach does not result in the identification of dynamic parameters of the system and only the lumped influence on the plant dynamics is estimated in the form of δ. Furthermore, the asymptotic convergence of $\tilde{\mathbf{z}}$ is not possible for varying $\boldsymbol{\psi}(t,\mathbf{x})$.

2.4. Gradient Approach for Parameter Estimation

Taking into account the estimation of the parameters of the system (6) one can employ the classic gradient algorithm [1,5,55] assuming that $\mathbf{x}$ and $\dot{\mathbf{x}}$ are available for measurement or can be otherwise obtained. From the dynamics (6) the following can be written

$$\dot{x}_n = bu + \boldsymbol{\psi}(t,\mathbf{x})\boldsymbol{\theta}, \tag{13}$$

which can be estimated by

$$\hat{\dot{x}}_n = bu + \boldsymbol{\psi}(t,\mathbf{x})\hat{\boldsymbol{\theta}}, \tag{14}$$

where $\hat{\boldsymbol{\theta}}$ is some estimate of $\boldsymbol{\theta}$. If $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, then clearly $\hat{\dot{x}}_n = \dot{x}_n$. If this is not the case, then an estimation error is expressed as

$$\tilde{\dot{x}}_n(t) = \dot{x}_n - \hat{\dot{x}}_n = \boldsymbol{\psi}(t,\mathbf{x})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) = \boldsymbol{\psi}(t,\mathbf{x})\tilde{\boldsymbol{\theta}}, \tag{15}$$

where $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ is the parameter identification error. Let $\tilde{\dot{x}}_n^2(t)$ be taken as a performance criterion to be minimized. The parameters of the model can then be modified according to the steepest descent of the performance function by an adaptation law proposed as

$$\dot{\hat{\boldsymbol{\theta}}} = -\frac{1}{2}\boldsymbol{\Gamma}\frac{d}{d\hat{\boldsymbol{\theta}}}\tilde{\dot{x}}_n^2(t) = \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t,\mathbf{x})\tilde{\dot{x}}_n(t), \tag{16}$$

where $\boldsymbol{\Gamma} = \text{diag}(\gamma_1,\dots,\gamma_k) \in \mathbb{R}^{k\times k}$ is some constant diagonal matrix of positive adaptation gains. It can be shown [1,56,57] that if Assumption 3 is satisfied, then under the adaptation law (16), the parameter estimate $\hat{\boldsymbol{\theta}}$ converges to the true value of $\boldsymbol{\theta}$. In particular, following the footsteps of [31], one can consider the following.

Corollary 1 (Persistently Excited Integral)**.** *If Assumptions 2 and 3 hold, then the following matrix*

$$\mathbf{M}(t,\mathbf{x}) = \int_t^\infty e^{(t-\tau)}\boldsymbol{\psi}^T(\tau,\mathbf{x}(\tau))\boldsymbol{\psi}(\tau,\mathbf{x}(\tau))d\tau \tag{17}$$

satisfies

$$\mu e^{-T}\|\mathbf{v}\|^2 \le \mathbf{v}^T\mathbf{M}\mathbf{v} \le \psi_{\text{M}}^2\|\mathbf{v}\|^2 \tag{18}$$

for any vector $\mathbf{v} \in \mathbb{R}^k$. *The time derivative of* $\mathbf{M}(t, \mathbf{x})$ *is given as*

$$\frac{d}{dt}\mathbf{M}(t,\mathbf{x}) = \int_t^\infty e^{(t-\tau)}\boldsymbol{\psi}^T(\tau,\mathbf{x}(\tau))\boldsymbol{\psi}(\tau,\mathbf{x}(\tau))d\tau - \boldsymbol{\psi}^T(t,\mathbf{x}(t))\boldsymbol{\psi}(t,\mathbf{x}(t)). \tag{19}$$

Lemma 1. *Given Assumptions 2 and 3 are satisfied, and the adaptation gains are chosen small enough, namely the following bound is satisfied* $\gamma_{\mathrm{M}} < \frac{1}{2}\psi_{\mathrm{M}}^{-2}$*, the identification law (16) guarantees the global asymptotic convergence of* $\hat{\boldsymbol{\theta}}$ *to the real parameters of system (6).*

Proof of Lemma 1. Recall Corollary 1 and consider a scalar function

$$V_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\mathbf{M}(t,\mathbf{x})\tilde{\boldsymbol{\theta}}. \tag{20}$$

Under Assumptions 2 and 3, the function $V_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}})$ satisfies

$$V_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}}) \geq \left(\frac{1}{2}\gamma_{\mathrm{M}}^{-1} - \psi_{\mathrm{M}}^2\right)\|\tilde{\boldsymbol{\theta}}\|^2, \tag{21}$$

namely, is positive definite for $\gamma_{\mathrm{M}} < \frac{1}{2}\psi_{\mathrm{M}}^{-2}$. Calculating the time derivative of $V_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}})$ one has

$$\dot{V}_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}}) = -\tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\boldsymbol{\psi}\tilde{\boldsymbol{\theta}} + 2\tilde{\boldsymbol{\theta}}^T\mathbf{M}\boldsymbol{\Gamma}\boldsymbol{\psi}^T\boldsymbol{\psi}\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\mathbf{M}\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\boldsymbol{\psi}\tilde{\boldsymbol{\theta}} \tag{22}$$

with the arguments of $\mathbf{M}(t,\mathbf{x})$ and $\boldsymbol{\psi}(t,\mathbf{x})$ omitted for brevity. This can be rewritten as

$$\dot{V}_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}}) = \tilde{\boldsymbol{\theta}}^T\left(2\mathbf{M}\boldsymbol{\Gamma}\boldsymbol{\psi}^T\boldsymbol{\psi} - \mathbf{M}\right)\tilde{\boldsymbol{\theta}}. \tag{23}$$

Although the first term in the bracket of (23) is not symmetric, it is guaranteed to be a bounded square matrix, and thus the whole multiplication results in a scalar value, which can in turn be upper-bounded by recalling a Cauchy–Schwarz inequality. Thus, the following bound of $\dot{V}_{\mathrm{grad}}$ can be established

$$\dot{V}_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}}) \leq \left(2\gamma_{\mathrm{M}}\psi_{\mathrm{M}}^4 - \mu e^{-T}\right)\|\tilde{\boldsymbol{\theta}}\|^2. \tag{24}$$

Hence, $\forall\tilde{\boldsymbol{\theta}} \neq \mathbf{0}$, $\dot{V}_{\mathrm{grad}} < 0$ for $\gamma_{\mathrm{M}} < \frac{1}{2}\psi_{\mathrm{M}}^{-4}\mu e^{-T}$. Thus, $V_{\mathrm{grad}}(\tilde{\boldsymbol{\theta}})$ is strictly positive definite with a strictly negative definite that proves a global asymptotic convergence of $\tilde{\boldsymbol{\theta}}$ to the origin. □

While Lemma 1 is more restrictive than the theorems given in the literature (as a specific choice of $\boldsymbol{\Gamma}$ is required, which is not a necessary according to other theoretical results, cf. [1]) and its proof closely follows the path of [31], according to the Authors' best knowledge, it has not yet been given in the literature in this form and is a basis of analysis conducted in further parts of this paper.

3. Parameter Identifying ESO

Multiple algorithms, known as adaptive observers, capable of simultaneous state and parameter estimation, have been proposed in the literature. Here, a novel approach employing Parameter Identifying ESO (PIESO), which combines classical ESO enhanced by the gradient parameter estimation, is proposed. Specifically, two distinct cases are considered, with a regressor given by $\boldsymbol{\psi}(t)$ and directly dependent on time only, or in the form of $\boldsymbol{\psi}(t,\mathbf{x})$ which explicitly depends on the unmeasurable state of the system.

3.1. State Independent Regressor

Consider a simplified version of the nominal system (6) given as

$$\dot{\mathbf{x}} = \mathbf{A}_n\mathbf{x} + \mathbf{b}_n bu + \mathbf{b}_n\boldsymbol{\psi}(t)\boldsymbol{\theta}, \tag{25}$$

with the regressor $\boldsymbol{\psi}(t)$ being a function of time and $\mathbf{A}_n, \mathbf{b}_n$ as defined in Section 2.1. Such definition of the system implies that the exact value of $\boldsymbol{\psi}(t)$ is known at any time instant and can be explicitly used in an identification procedure. To estimate the state and parameters of this system, the following extended dynamics can be considered

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_n\mathbf{x} + \mathbf{b}_n\big(bu + \boldsymbol{\psi}(t)\hat{\boldsymbol{\theta}} + \delta\big), \\ \dot{\delta} &= \frac{d}{dt}\big(\boldsymbol{\psi}(t)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big)\big), \end{aligned} \tag{26}$$

where $\hat{\boldsymbol{\theta}}$ stands for some estimate of parameter $\boldsymbol{\theta}$, and $\delta = \boldsymbol{\psi}(t)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big)$ is the total disturbance corresponding to the modeling error. Clearly, if $\hat{\boldsymbol{\theta}} = \boldsymbol{\theta}$, then $\delta = 0$ for any time instant. On the contrary, if the value of $\boldsymbol{\theta}$ is not known, then δ directly corresponds to the estimation error defined in Section 2.4 as (15). This notion can be used to synthesize the adaptive Parameter Identifying ESO. Introducing the extended state $\mathbf{z} = \begin{bmatrix}\mathbf{x}^T & \delta\end{bmatrix}^T \in \mathbb{R}^m$ one can rewrite (26) as

$$\dot{\mathbf{z}} = \mathbf{A}_m\mathbf{z} + \mathbf{d}_m\big(bu + \boldsymbol{\psi}(t)\hat{\boldsymbol{\theta}}\big) + \mathbf{b}_m\frac{d}{dt}\boldsymbol{\psi}(t)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big). \tag{27}$$

For the system formulated in this way, the PIESO can be proposed as

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}_m\hat{\mathbf{z}} + \mathbf{d}_m\big(bu + \boldsymbol{\psi}(t)\hat{\boldsymbol{\theta}}\big) + \mathbf{l}(z_1 - \hat{z}_1), \tag{28}$$

with the parameter adaptation law given by

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\hat{z}_m, \tag{29}$$

where $\mathbf{l} = \begin{bmatrix} l_1 & \dots & l_m \end{bmatrix}^T \in \mathbb{R}_+^m$ are positive observer gains chosen to ensure the stability of the observer and $\boldsymbol{\Gamma} = \mathrm{diag}(\gamma_1, \dots, \gamma_k) \in \mathbb{R}_+^{k\times k}$ is a positive diagonal adaptation gain matrix. Consider the observer tuning $l_i = \bar{l}_i\omega_o^i$ where $\omega_o \in \mathbb{R}_+$ is called the observer bandwidth and is considered as a new tuning variable [58]. The theorem regarding the stability of the system can now be formulated.

Theorem 1. *Given Assumptions 2 and 3 are satisfied, then, for the system (25), the observer (28) with the adaptation law (29), guarantees a global asymptotic convergence of $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\theta}}$ to $\mathbf{z}$ and $\boldsymbol{\theta}$ correspondingly, if $\bar{\mathbf{l}}$ gains are chosen such that the matrix $\bar{\mathbf{H}} = \mathbf{A}_m - \bar{\mathbf{l}}\mathbf{c}_m^T$ is Hurwitz, the parameter ω_o is chosen high enough and $\boldsymbol{\Gamma}$ is set small enough.*

Proof of Theorem 1. To prove Theorem 1 consider the estimation and identification errors $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$ and $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. By recalling (27) and (28) with (29) and remembering that $\delta = \boldsymbol{\psi}(t)\big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\big)$, the errors dynamics are given by

$$\begin{aligned} \dot{\tilde{\mathbf{z}}} &= \mathbf{H}\tilde{\mathbf{z}} + \mathbf{b}_m\dot{\boldsymbol{\psi}}(t)\tilde{\boldsymbol{\theta}} - \boldsymbol{\psi}(t)\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m\boldsymbol{\psi}(t)\tilde{\boldsymbol{\theta}} + \boldsymbol{\psi}(t)\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m\mathbf{b}_m^T\tilde{\mathbf{z}}, \\ \dot{\tilde{\boldsymbol{\theta}}} &= -\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\boldsymbol{\psi}(t)\tilde{\boldsymbol{\theta}} + \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m^T\tilde{\mathbf{z}}, \end{aligned} \tag{30}$$

where $\mathbf{H} = \mathbf{A}_m - \mathbf{l}\mathbf{c}_m^T$. Consider the scaled estimation errors as $\bar{\mathbf{z}} = \boldsymbol{\Phi}_m(\omega_o)\tilde{\mathbf{z}}$, which can be used to express the error dynamics as

$$\begin{aligned} \dot{\bar{\mathbf{z}}} &= \omega_o\bar{\mathbf{H}}\bar{\mathbf{z}} + \mathbf{b}_m\dot{\boldsymbol{\psi}}(t)\tilde{\boldsymbol{\theta}} - \boldsymbol{\psi}(t)\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m\boldsymbol{\psi}(t)\tilde{\boldsymbol{\theta}} + \boldsymbol{\psi}(t)\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m\mathbf{b}_m^T\bar{\mathbf{z}}, \\ \dot{\tilde{\boldsymbol{\theta}}} &= -\boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\boldsymbol{\psi}(t)\tilde{\boldsymbol{\theta}} + \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t)\mathbf{b}_m^T\bar{\mathbf{z}}, \end{aligned} \tag{31}$$

with $\bar{\mathbf{H}} = \mathbf{A}_m - \bar{\mathbf{l}}\mathbf{c}_m^T$ as in Theorem 1. Note that for properly chosen $\bar{\mathbf{l}}$ the matrix $\bar{\mathbf{H}}$ is Hurwitz and satisfies $\bar{\mathbf{H}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{H}} = -\mathbf{Q}$ for any positive definite matrix $\mathbf{Q} \in \mathbb{R}^{m\times m}$ and some positive definite matrix $\mathbf{P} \in \mathbb{R}^{m\times m}$. Recalling Assumptions 2 and 3 with Corollary 1 the following function can be considered for the system with time dependent regressor,

$$V_{\rm t}(\bar{\mathbf{z}},\tilde{\theta}) = \frac{1}{2}\bar{\mathbf{z}}^T\mathbf{P}\bar{\mathbf{z}} + \frac{1}{2}\tilde{\theta}^T\mathbf{\Gamma}^{-1}\tilde{\theta} - \tilde{\theta}^T\mathbf{M}(t)\tilde{\theta}. \tag{32}$$

Clearly, the function $V_{\rm t}(\bar{\mathbf{z}},\tilde{\theta})$ satisfies

$$V_{\rm t}(\bar{\mathbf{z}},\tilde{\theta}) \geq \frac{1}{2}p_{\rm m}\|\bar{\mathbf{z}}\|^2 + \left(\frac{1}{2}\gamma_{\rm M}^{-1} - \psi_{\rm M}^2\right)\|\tilde{\theta}\|^2 \tag{33}$$

and is positive definite for any $\gamma_{\rm M} < \frac{1}{2}\psi_{\rm M}^{-2}$. The time derivative of $V_{\rm t}(\bar{\mathbf{z}},\tilde{\theta})$ is given by

$$\begin{aligned}\dot{V}_{\rm t}(\bar{\mathbf{z}},\tilde{\theta}) = {} & \bar{\mathbf{z}}^T\left(\mathbf{P}\psi\mathbf{\Gamma}\psi^T\mathbf{b}_m\mathbf{b}_m^T - \frac{1}{2}\omega_{\rm o}\mathbf{Q}\right)\bar{\mathbf{z}} + \tilde{\theta}^T\left(2\mathbf{M}\mathbf{\Gamma}\psi^T\psi - \mathbf{M}\right)\tilde{\theta} \\ & + \bar{\mathbf{z}}^T\left(\mathbf{P}\mathbf{b}_m\dot{\psi} - \mathbf{P}\psi\mathbf{\Gamma}\psi^T\mathbf{b}_m\psi + \mathbf{b}_m\psi - 2\mathbf{b}_m\psi\mathbf{\Gamma}\mathbf{M}\right)\tilde{\theta},\end{aligned} \tag{34}$$

with arguments of $\psi(t)$ and $\mathbf{M}(t)$ again omitted for brevity. The derivative of $V_{\rm t}(\bar{\mathbf{z}},\tilde{\theta})$ satisfies

$$\begin{aligned}\dot{V}_{\rm t}(\bar{\mathbf{z}},\tilde{\theta}) \leq {} & \left(\frac{1}{2}\epsilon\left(p_{\rm M}\psi_{\rm M} + \psi_{\rm M} + \gamma_{\rm M}p_{\rm M}\psi_{\rm M}^3 + \gamma_{\rm M}\psi_{\rm M}^3\right)^2 + p_{\rm M}\psi_{\rm M}^2\gamma_{\rm M} - \omega_{\rm o}q_{\rm m}\right)\|\bar{\mathbf{z}}\|^2 \\ & + \left(\frac{1}{2\epsilon} + 2\psi_{\rm M}^4\gamma_{\rm M} - \mu e^{-T}\right)\|\tilde{\theta}\|^2,\end{aligned} \tag{35}$$

for any $\epsilon \in \mathbb{R}_+$. Set now $\mathbf{\Gamma}$ such that $\gamma_{\rm M} < \frac{1}{2}\psi_{\rm M}^{-4}\mu e^{-T}$ what makes the second term negative with $\epsilon > \frac{1}{2}\left(\mu e^{-T} - 2\psi_{\rm M}^4\gamma_{\rm M}\right)^{-1}$. Choosing $\omega_{\rm o}$ high enough now yields the negative definiteness of the whole $\dot{V}_{\rm t}(\bar{\mathbf{z}},\tilde{\theta})$ that concludes the proof. □

While Theorem 1 states that adaptation gains $\mathbf{\Gamma}$ have to be chosen smaller than some specific values dependent on the dynamics of $\psi(t)$, this proof closely follows the proof of Lemma 1 which is known to be conservative. Thus, it is justified to believe that this limitation on $\mathbf{\Gamma}$ may be lifted with a better suited proof. If this is the case, it could be expected that only a requirement concerning the value of $\omega_{\rm o}$ would be formulated depending on the characteristics of $\dot{\psi}(t)$, cf. (4).

To make the proposed identification procedure clearer, the corresponding block diagram is presented in Figure 2.

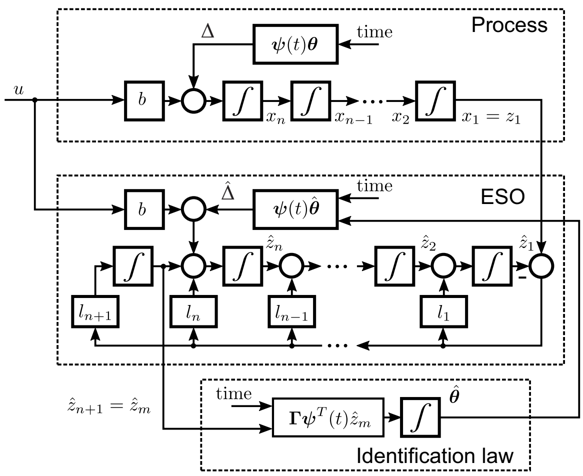

Figure 2. Identification block diagram for the state independent regressor.

3.2. State Dependent Regressor

The algorithm proposed in Section 3.1 offers a simple and intuitive solution to the problem of simultaneous state and parameter estimation, but its applicability is limited only to the systems with exactly known regressor independent of the state of the plant. To overcome this limitation, an enhanced algorithm has to be formulated for a wider class of dynamic systems. Consider once again the system (6) in its nominal form, with $\boldsymbol{\psi}(t, \mathbf{x})$ depending on both the time and the state of the system, given by

$$\dot{\mathbf{x}} = \mathbf{A}_n \mathbf{x} + \mathbf{b}_n b u + \mathbf{b}_n \boldsymbol{\psi}(t, \mathbf{x}) \boldsymbol{\theta}. \tag{36}$$

Note that (36) becomes a standard linear time-invariant system commonly considered in the literature on parameter identification if $\boldsymbol{\psi}(t, \mathbf{x}) = \mathbf{x}^T$. Thus, the proposed observer can be applied to the LTI system if its evolution is such that $\mathbf{x}$ and $\dot{\mathbf{x}}$ are both bounded to ensure that Assumption 2 holds. The system (36) can be once again expressed using an extended state $\mathbf{z} = \begin{bmatrix} \mathbf{x}^T & \delta \end{bmatrix}^T \in \mathbb{R}^m$

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_n \mathbf{x} + \mathbf{b}_n \big(bu + \boldsymbol{\psi}(t, \mathbf{x}) \hat{\boldsymbol{\theta}} + \delta \big), \\ \dot{\delta} &= \frac{d}{dt} \big(\boldsymbol{\psi}(t, \mathbf{x}) \big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \big) \big), \end{aligned} \tag{37}$$

where $\hat{\boldsymbol{\theta}}$ denotes some estimate of the parameters $\boldsymbol{\theta}$, and $\delta = \boldsymbol{\psi}(t, \mathbf{x}) \big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \big)$ is a total disturbance corresponding to the modeling error. Notice that the total disturbance is here formulated using a state $\mathbf{x}$ of the system, which is, in general, unknown. Rewrite (37) as

$$\dot{\mathbf{z}} = \mathbf{A}_m \mathbf{z} + \mathbf{d}_m \big(bu + \boldsymbol{\psi}(t, \boldsymbol{\Lambda} \mathbf{z}) \hat{\boldsymbol{\theta}} \big) + \mathbf{b}_m \frac{d}{dt} \big(\boldsymbol{\psi}(t, \boldsymbol{\Lambda} \mathbf{z}) \big(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \big) \big), \tag{38}$$

where $\boldsymbol{\Lambda} \mathbf{z} = \mathbf{x}$. For the system formulated in such a way, one can propose the following PIESO

$$\dot{\hat{\mathbf{z}}} = \mathbf{A}_m \hat{\mathbf{z}} + \mathbf{d}_m \big(bu + \boldsymbol{\psi}(t, \boldsymbol{\Lambda} \hat{\mathbf{z}}) \hat{\boldsymbol{\theta}} \big) + \mathbf{l} (z_1 - \hat{z}_1), \tag{39}$$

which does not require knowledge of the state $\mathbf{x}$ to calculate the regressor. The term $\mathbf{l} = \begin{bmatrix} l_1 & \dots & l_m \end{bmatrix}^T \in \mathbb{R}_+^m$ again represents positive observer gains. The parameter adaptation law is proposed as

$$\dot{\hat{\boldsymbol{\theta}}} = \operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta}), \quad \boldsymbol{\tau} = \boldsymbol{\Gamma} \boldsymbol{\psi}^T(t, \boldsymbol{\Lambda} \hat{\mathbf{z}}) \hat{z}_m, \tag{40}$$

where $\boldsymbol{\Gamma} = \operatorname{diag}(\gamma_1, \dots, \gamma_k) \in \mathbb{R}_+^{k \times k}$ is a diagonal matrix of positive adaptation gains. The term $\operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ stands for a projection operator [2,42,59] which satisfies

$$\tilde{\boldsymbol{\theta}} \boldsymbol{\Gamma}^{-1} \big(\operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta}) - \boldsymbol{\tau} \big) \geq 0, \tag{41}$$

and $\hat{\boldsymbol{\theta}} \in \boldsymbol{\Theta}$, with $\boldsymbol{\Theta}$ being some predefined convex set containing real values of $\boldsymbol{\theta}$. Throughout the rest of this paper, the second and third parameter of $\operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ will be omitted for brevity.

Remark 2. *Multiple propositions of* $\operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ *are given in the literature [60], with one of the most common choices being elementwise operator*

$$\operatorname{Proj}_i(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta}) = \begin{cases} 0 & \tau_i > 0 \wedge \hat{\theta}_i \geq \theta_{iM} \\ 0 & \tau_i < 0 \wedge \hat{\theta}_i \leq -\theta_{iM} \\ \tau_i & \text{otherwise} \end{cases} \tag{42}$$

with the convex set defined as $\boldsymbol{\Theta} = \{\boldsymbol{\theta} : \theta_i \in (-\theta_{iM}, \theta_{iM}), i = 1, \dots, k\}$. *The analysis presented further does not assume any specific choice of* $\operatorname{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})$ *operator.*

Corollary 2 (Boundedness of identification errors)**.** *Directly from* (41) *and the definition of the projection operator, it can be shown that there exists a constant* $\theta_{\mathrm{M}} \in \mathbb{R}_+$ *such that*

$$\max(\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|_\infty, \|\hat{\boldsymbol{\theta}}\|_\infty) \leq \theta_{\mathrm{M}}. \tag{43}$$

Thus, it can be stated that $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|^2 \leq \|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|\theta_{\mathrm{M}}$. *Moreover,* $\|\mathrm{Proj}(\boldsymbol{\tau}, \hat{\boldsymbol{\theta}}, \boldsymbol{\Theta})\| \leq \|\boldsymbol{\tau}\|$ *for any* $\boldsymbol{\tau}$.

Remark 3. *Note that the bound* θ_{M} *which is imposed on the estimation error* $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}\|$ *according to Corollary 2 holds even if* $\hat{\boldsymbol{\theta}}(0) \notin \boldsymbol{\Theta}$. *In such a case, the bound* θ_{M} *does not come directly from the size of the set* $\boldsymbol{\Theta}$ *but corresponds to the* $\|\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(0)\|$ *which will never grow larger due to the projection operator. Thus, the estimation error remains bounded and the further analysis holds in full.*

Once again, consider the observer tuning $l_i = \bar{l}_i\omega_{\mathrm{o}}^i$ where $\omega_{\mathrm{o}} \in \mathbb{R}_+$ is the observer bandwidth. Under such tuning, the theorem concerning the stability of the PIESO with the regressor explicitly depending on the state of the system can be formulated.

Theorem 2. *Given Assumptions 2 and 3 are satisfied, the observer* (39) *along with the adaptation law* (40) *guarantees the global asymptotic convergence of* $\hat{\mathbf{z}}$ *and* $\hat{\boldsymbol{\theta}}$ *to the extended state trajectory* $\mathbf{z}$ *and the parameters* $\boldsymbol{\theta}$ *of the system* (36)*, respectively, if gains* $\bar{\mathbf{l}}$ *are selected such that the matrix* $\bar{\mathbf{H}} = \mathbf{A}_m - \bar{\mathbf{l}}\mathbf{c}_m^T$ *is Hurwitz, the parameter* ω_{o} *is chosen high enough and* $\boldsymbol{\Gamma}$ *is set small enough.*

Proof of Theorem 2. Consider the estimation errors $\tilde{\mathbf{z}} = \mathbf{z} - \hat{\mathbf{z}}$ and $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ with dynamics given as

$$\begin{aligned} \dot{\tilde{\mathbf{z}}} &= \mathbf{H}\tilde{\mathbf{z}} + \mathbf{d}_m(\boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z}) - \boldsymbol{\psi}(t, \boldsymbol{\Lambda}\hat{\mathbf{z}}))\hat{\boldsymbol{\theta}} + \mathbf{b}_m\frac{d}{dt}(\boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z})\tilde{\boldsymbol{\theta}}), \\ \dot{\tilde{\boldsymbol{\theta}}} &= -\mathrm{Proj}(\boldsymbol{\tau}), \quad \boldsymbol{\tau} = \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t, \boldsymbol{\Lambda}\hat{\mathbf{z}})\tilde{z}_m, \end{aligned} \tag{44}$$

where $\mathbf{H} = \mathbf{A}_m - \mathbf{l}\mathbf{c}_m^T$. Consider again the scaled estimation errors $\bar{\mathbf{z}} = \boldsymbol{\Phi}_m(\omega_{\mathrm{o}})\tilde{\mathbf{z}}$, with dynamics

$$\begin{aligned} \dot{\bar{\mathbf{z}}} &= \omega_{\mathrm{o}}\bar{\mathbf{H}}\bar{\mathbf{z}} + \omega_{\mathrm{o}}\mathbf{d}_m(\boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z}) - \boldsymbol{\psi}(t, \boldsymbol{\Lambda}\hat{\mathbf{z}}))\hat{\boldsymbol{\theta}} + \mathbf{b}_m\frac{d}{dt}(\boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z})\tilde{\boldsymbol{\theta}}), \\ \dot{\tilde{\boldsymbol{\theta}}} &= -\mathrm{Proj}(\boldsymbol{\tau}), \quad \boldsymbol{\tau} = \boldsymbol{\Gamma}\boldsymbol{\psi}^T(t, \boldsymbol{\Lambda}\hat{\mathbf{z}})\tilde{z}_m, \end{aligned} \tag{45}$$

where $\bar{\mathbf{H}} = \mathbf{A}_m - \bar{\mathbf{l}}\mathbf{c}_m^T$ is Hurwitz and satisfies $\bar{\mathbf{H}}^T\mathbf{P} + \mathbf{P}\bar{\mathbf{H}} = -\mathbf{Q}$ for any positive definite matrix $\mathbf{Q} \in \mathbb{R}^{m \times m}$ and some positive definite matrix $\mathbf{P} \in \mathbb{R}^{m \times m}$. Further analysis of the estimation stability will be done in two distinct steps. First, under the notion of the boundedness of identification errors guaranteed by the projection operator, the boundedness of state estimates will be established. Next, this notion will be used to conclude an asymptotic convergence of both identification and estimation errors to the origin. Consider the function

$$V_{\mathrm{z}}(\bar{\mathbf{z}}) = \frac{1}{2}\bar{\mathbf{z}}^T\mathbf{P}\bar{\mathbf{z}} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\mathbf{M}(t, \boldsymbol{\Lambda}\mathbf{z})\tilde{\boldsymbol{\theta}} - \bar{\mathbf{z}}^T\mathbf{P}\mathbf{b}_m\boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z})\tilde{\boldsymbol{\theta}}, \tag{46}$$

which satisfies

$$V_{\mathrm{z}}(\bar{\mathbf{z}}) \geq \left(\frac{1}{2}p_{\mathrm{m}} - \frac{1}{2\epsilon_0}p_{\mathrm{M}}\psi_{\mathrm{M}}^2\right)\|\bar{\mathbf{z}}\|^2 + \left(\frac{1}{2}\gamma_{\mathrm{M}}^{-1} - \psi_{\mathrm{M}}^2 - \frac{\epsilon_0}{2}p_{\mathrm{M}}\psi_{\mathrm{M}}^2\right)\|\tilde{\boldsymbol{\theta}}\|^2, \tag{47}$$

for any $\epsilon_0 \in \mathbb{R}_+$. The function $V_{\mathrm{z}}(\bar{\mathbf{z}})$ is clearly positive for any choice of $\boldsymbol{\Gamma}$ that satisfies $\gamma_{\mathrm{M}} < \frac{1}{2}\left(\boldsymbol{\psi}_{\mathrm{M}}^2 + \frac{\epsilon_0}{2}p_{\mathrm{M}}\boldsymbol{\psi}_{\mathrm{M}}^2\right)^{-1}$ with $\epsilon_0 > p_{\mathrm{m}}^{-1}p_{\mathrm{M}}\boldsymbol{\psi}_{\mathrm{M}}^2$. The time derivative of $V_{\mathrm{z}}(\bar{\mathbf{z}})$ is given by

$$\dot{V}_{\rm z}(\bar{\mathbf{z}}) = \omega_{\rm o}\bar{\mathbf{z}}^T\mathbf{P}\bar{\mathbf{H}}\bar{\mathbf{z}} + \omega_{\rm o}\bar{\mathbf{z}}^T\mathbf{P}\mathbf{d}_m\tilde{\boldsymbol{\psi}}\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\operatorname{Proj}(\boldsymbol{\tau}) + 2\tilde{\boldsymbol{\theta}}^T\mathbf{M}\operatorname{Proj}(\boldsymbol{\tau}) - \tilde{\boldsymbol{\theta}}^T\mathbf{M}\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\boldsymbol{\psi}\tilde{\boldsymbol{\theta}}$$
$$-\omega_{\rm o}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\mathbf{b}_m^T\mathbf{P}\bar{\mathbf{H}}\bar{\mathbf{z}} - \omega_{\rm o}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\mathbf{b}_m^T\mathbf{P}\mathbf{d}_m\tilde{\boldsymbol{\psi}}\hat{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\mathbf{b}_m^T\mathbf{P}\mathbf{b}_m\dot{\boldsymbol{\psi}}\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\mathbf{b}_m^T\mathbf{P}\mathbf{b}_m\boldsymbol{\psi}\operatorname{Proj}(\boldsymbol{\tau}), \quad (48)$$

where $\boldsymbol{\psi} = \boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z})$ and $\tilde{\boldsymbol{\psi}} = \boldsymbol{\psi}(t, \boldsymbol{\Lambda}\mathbf{z}) - \boldsymbol{\psi}(t, \boldsymbol{\Lambda}\hat{\mathbf{z}})$ are denoted for brevity. By recalling Corollary 2 and Assumption 2, the derivative $\dot{V}_{\rm z}(\bar{\mathbf{z}})$ can be bounded by

$$\dot{V}_{\rm z}(\bar{\mathbf{z}}) \leq \left(-\frac{1}{2}\omega_{\rm o}q_{\rm m} + \left\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\right\|\left(\omega_{\rm o}p_{\rm M}\psi_{\rm M}\theta_{\rm M} + \theta_{\rm M}\psi_{\rm M} + 2\theta_{\rm M}\psi_{\rm M}^3\gamma_{\rm M} + \theta_{\rm M}\psi_{\rm M}^3p_{\rm M}\gamma_{\rm M}\right)\right)\|\bar{\mathbf{z}}\|^2$$
$$+\left\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\right\|\left(\psi_{\rm M}^2\theta_{\rm M}^2 + 2\theta_{\rm M}^2\psi_{\rm M}^4\gamma_{\rm M} + \omega_{\rm o}\theta_{\rm M}^2\psi_{\rm M}^2p_{\rm M} + \theta_{\rm M}^2\psi_{\rm M}^4p_{\rm M}\gamma_{\rm M}\right)\|\bar{\mathbf{z}}\|$$
$$+\left(\theta_{\rm M}\psi_{\rm M} + 2\theta_{\rm M}\psi_{\rm M}^3\gamma_{\rm M} + \omega_{\rm o}\tilde{\theta}_{\rm M}\psi_{\rm M}p_{\rm M}h_{\rm M} + \theta_{\rm M}\psi_{\rm M}^3p_{\rm M}\gamma_{\rm M}\right)\|\bar{\mathbf{z}}\|$$
$$+2\theta_{\rm M}^2\psi_{\rm M}^4\gamma_{\rm M} - \mu e^{-T}\theta_{\rm M}^2 + 2\psi_{\rm M}^2\theta_{\rm M}^2 + \theta_{\rm M}^2\psi_{\rm M}^2p_{\rm M} + \theta_{\rm M}^2\psi_{\rm M}^4p_{\rm M}\gamma_{\rm M}, \quad (49)$$

where $\boldsymbol{\Phi} = \boldsymbol{\Phi}_m(\omega_{\rm o})$ is denoted for brevity. It can be noticed that both $\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\|$ and $\omega_{\rm o}\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\|$ are nonincreasing in $\omega_{\rm o}$ for any $\omega_{\rm o} \geq 1$. Thus, there exists $\omega_{\rm o}$ high enough to ensure negativeness of $\dot{V}_{\rm z}(\bar{\mathbf{z}})$ for $\|\bar{\mathbf{z}}\|$ large enough and the convergence of $\bar{\mathbf{z}}$ to some boundary of origin is achieved. Consequently, it can be concluded that there exists $z_{\rm M}, T_z \in \mathbb{R}_+$ such that $\|\bar{\mathbf{z}}\| \leq z_{\rm M}$ for any $t \geq T_z$.

Having established the notion of boundedness of $\bar{\mathbf{z}}$ for $t \geq T_z$, one can consider the following function

$$V_{\rm tx}(\bar{\mathbf{z}}, \tilde{\boldsymbol{\theta}}) = \frac{1}{2}\bar{\mathbf{z}}^T\mathbf{P}\bar{\mathbf{z}} + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\mathbf{M}(t, \boldsymbol{\Lambda}\mathbf{z})\tilde{\boldsymbol{\theta}}, \quad (50)$$

which can be bounded by

$$V_{\rm tx}(\bar{\mathbf{z}}) \geq \frac{1}{2}p_{\rm m}\|\bar{\mathbf{z}}\|^2 + \left(\frac{1}{2}\gamma_{\rm M}^{-1} - \psi_{\rm M}^2\right)\|\tilde{\boldsymbol{\theta}}\|^2. \quad (51)$$

Hence, one can show that $V_{\rm tx}$ is positive definite for any $\gamma_{\rm M} < \frac{1}{2}\boldsymbol{\psi}_{\rm M}^{-2}$. Its derivative is given as

$$\dot{V}_{\rm tx}(\bar{\mathbf{z}}, \tilde{\boldsymbol{\theta}}) = -\frac{1}{2}\omega_{\rm o}\bar{\mathbf{z}}^T\mathbf{Q}\bar{\mathbf{z}} + \omega_{\rm o}\bar{\mathbf{z}}^T\mathbf{P}\mathbf{d}_m\tilde{\boldsymbol{\psi}}\hat{\boldsymbol{\theta}} + \bar{\mathbf{z}}^T\mathbf{P}\mathbf{b}_m\dot{\boldsymbol{\psi}}\tilde{\boldsymbol{\theta}} + \bar{\mathbf{z}}^T\mathbf{P}\mathbf{b}_m\boldsymbol{\psi}\operatorname{Proj}(\boldsymbol{\tau})$$
$$-\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\operatorname{Proj}(\boldsymbol{\tau}) + \tilde{\boldsymbol{\theta}}^T\mathbf{M}\operatorname{Proj}(\boldsymbol{\tau}) - \tilde{\boldsymbol{\theta}}^T\mathbf{M}\tilde{\boldsymbol{\theta}} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\psi}^T\boldsymbol{\psi}\tilde{\boldsymbol{\theta}}. \quad (52)$$

Now, by adding and subtracting the term $\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\boldsymbol{\tau}$ to (52) and then recalling Corollary 2, Assumption 2 and (41) it can be infered that

$$\dot{V}_{\rm tx}(\bar{\mathbf{z}}, \tilde{\boldsymbol{\theta}}) \leq \left(\frac{1}{2\epsilon_1} + \gamma_{\rm M}\psi_{\rm M}^4 - \mu e^{-T}\right)\|\tilde{\boldsymbol{\theta}}\|^2 + \left(-\frac{1}{2}\omega_{\rm o}q_{\rm m} + p_{\rm M}\gamma_{\rm M}\psi_{\rm M}^2\right)\|\bar{\mathbf{z}}\|^2$$
$$+\left\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\right\|\left(\omega_{\rm o}p_{\rm M}\psi_{\rm M}\theta_{\rm M} + p_{\rm M}\gamma_{\rm M}\psi_{\rm M}^3\theta_{\rm M} + z_{\rm M}p_{\rm M}\gamma_{\rm M}\psi_{\rm M}^3 + \theta_{\rm M}\psi_{\rm M} + \theta_{\rm M}\gamma_{\rm M}\psi_{\rm M}^4\right)\|\bar{\mathbf{z}}\|^2 \quad (53)$$
$$+\frac{\epsilon_1}{2}\left(p_{\rm M}\psi_{\rm M} + p_{\rm M}\gamma_{\rm M}\psi_{\rm M}^3 + \psi_{\rm M} + \gamma_{\rm M}\psi_{\rm M}^3 + \left\|\boldsymbol{\Lambda}\boldsymbol{\Phi}^{-1}\right\|\left(\psi_{\rm M}^2\theta_{\rm M} + \theta_{\rm M}\gamma_{\rm M}\psi_{\rm M}^4\right)\right)^2\|\bar{\mathbf{z}}\|^2$$

for any $\epsilon_1 \in \mathbb{R}_+$. Set $\boldsymbol{\Gamma}$ such that $\gamma_{\rm M} < \mu e^{-T}\psi_{\rm M}^{-4}$. Then there exists some ϵ_1 large enough such that for some $\omega_{\rm o}$ high enough, the whole $\dot{V}_{\rm tx}(\bar{\mathbf{z}}, \tilde{\boldsymbol{\theta}})$ is negative definite. Thus, once the estimation errors converge below the arbitrary threshold $z_{\rm M}$, the asymptotic convergence of the estimation and identification errors to the origin is guaranteed. This concludes the proof. □

The graphical interpretation of the discussed identification procedure is shown in Figure 3.

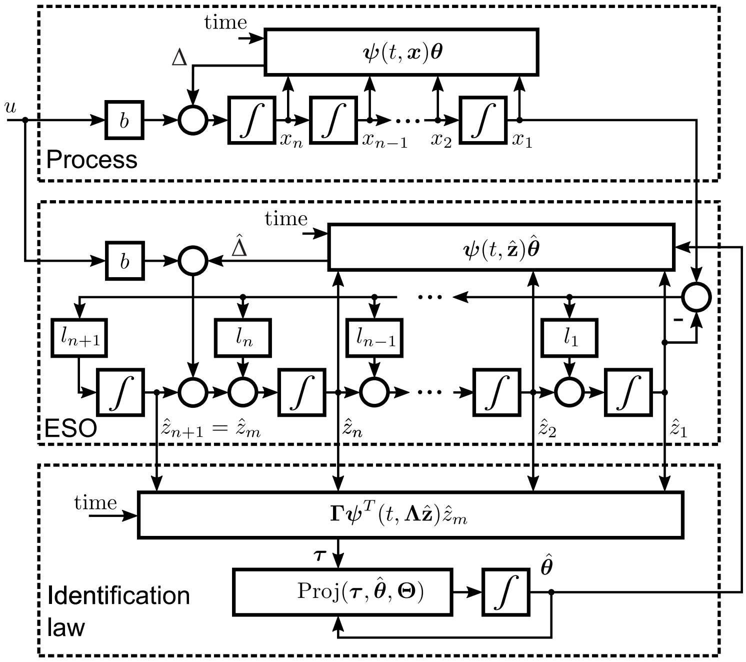

Figure 3. Identification block diagram for the state dependent regressor.

4. Exemplary Scenarios and Simulation Validation

In order to verify the usability of the proposed identification algorithm, a series of simulations have been performed. First, two nominal cases corresponding to Sections 3.1 and 3.2 with regressor depending on time and then on the time and the state are presented. Then, an additional example of a harmonic disturbance identification is presented as an illustration of the possible applicability of the considered scheme outside its nominal area. It has to be noted that the gains and parameters chosen for each simulation do not guarantee the fastest convergence speed achievable and were chosen in order to ease the evaluation of the presented results.

4.1. Nominal Cases

The initial test is performed for a simplified case of the regressor independent of the state of the system and depending on the time only. A simple second-order system ($n = 2$), consistent with (25), is chosen, with three unknown parameters ($k = 3$) to be identified based on the persistently exciting regressor. The first simulation is performed with the adaptation gain set to zero, and thus the proposed PIESO is reduced to a classic ESO observer. All parameters used in this simulation are given in Table 1.

Table 1. Parameters of the simulation of ESO without adaptation.

ω_o	Γ	$\psi(t)$	θ	$\mathbf{x}(0)$	$\hat{\mathbf{z}}(0)$	$\hat{\theta}(0)$	$bu(t)$
50	$\mathbf{0}$	$\begin{bmatrix}\sin^2(t)\\ \cos(3t)\\ 1\end{bmatrix}^T$	$\begin{bmatrix}2\\3\\4\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\end{bmatrix}$	$\sin(t)$

The regressor, which is here the only source of unknown disturbance δ that affects the plant, consists of two periodic harmonic functions and a single constant value representing an offset influencing the plant. The input signal is chosen as a sine wave, leading to an unbounded growth of state variables $\mathbf{x}$. The results of the first simulation are given in Figure 4 where the errors of estimation of the state vector $\mathbf{x}$ and the value of the total disturbance estimate are shown.

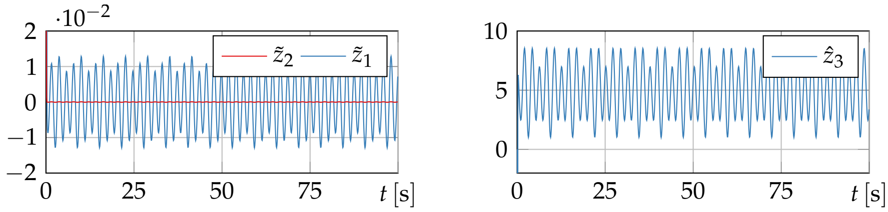

Figure 4. Results of the simulation without adaptation with disturbance depending on time only.

In accordance with the analysis given in Section 2.3, estimation errors converge only to some non-zero neighborhood of origin. The estimate of the total disturbance tracks the influence of the unknown dynamics on the behavior of the plant.

To evaluate the performance of the proposed algorithm, simulation of the same system with the adaptation enabled is repeated. All parameters of this trial are given in Table 2.

Table 2. Parameters of the simulation of PIESO with time-dependent regressor.

ω_o	Γ	$\psi(t)$	θ	$\mathbf{x}(0)$	$\hat{\mathbf{z}}(0)$	$\hat{\theta}(0)$	$bu(t)$
50	$0.1\mathbf{I}$	$\begin{bmatrix}\sin^2(t)\\ \cos(3t)\\ 1\end{bmatrix}^T$	$\begin{bmatrix}2\\3\\4\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\mathbf{0}_{3\times1}$	$\mathbf{0}_{3\times1}$	$\sin(t)$

The results of the simulation of PIESO with the regressor dependent only on the time are given in Figure 5. Here, the estimation errors and the total disturbance estimate are shown along with the identification errors, the eigenvalues of the integral $\mathbf{M}(t)$ and the function $V_t(\bar{\mathbf{z}}, \tilde{\theta})$ with its derivative.

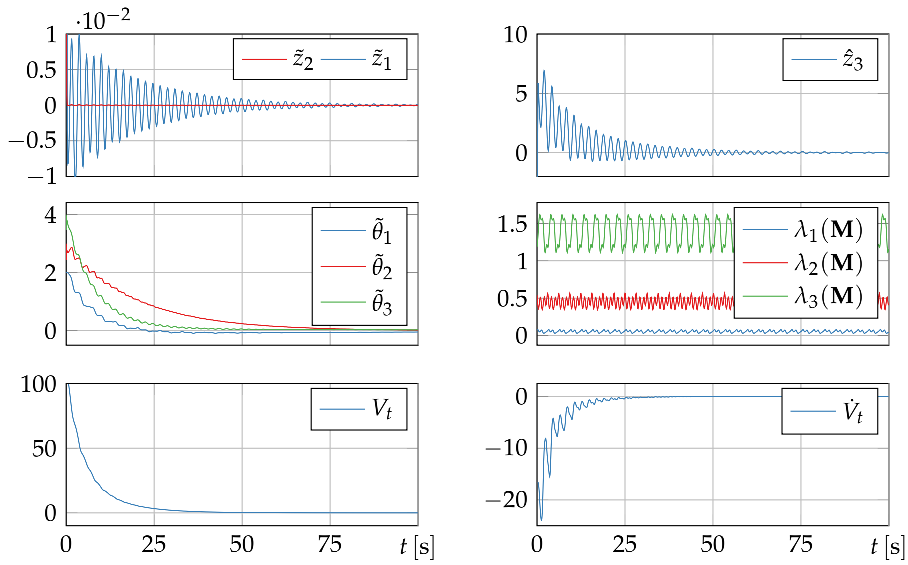

Figure 5. Results of the simulation with regressor depending on time only.

It can be seen that the proposed algorithm successfully identified all parameters of the system and guaranteed the asymptotic convergence of all errors to the origin, which is not achievable with ordinary ESO. Moreover, while the parameter estimation process takes approximately 50 s, the quality of the state estimation comparable to the standard ESO is achieved almost immediately, which constitutes a significant advantage of the proposed method in comparison with other adaptive observer designs. As expected, the estimate of the total disturbance vanishes with time, as the process of parameter identification progresses. The evolution of the function $V_t(\bar{\mathbf{z}}, \tilde{\theta})$ and $\dot{V}_t(\bar{\mathbf{z}}, \tilde{\theta})$ along with the positiveness of all eigenvalues of integral $\mathbf{M}(t)$ is consistent with the analysis presented in Section 3.1.

Correct estimation and identification results are obtained despite the unbounded evolution of the state **x** of the system itself as Assumption 2 requires only the boundedness of the regressor $\boldsymbol{\psi}(t)$, which in the considered case is independent on the system state. For comparison, the simulation was repeated with different choices of the adaptation gain matrix $\boldsymbol{\Gamma}$. The evolution of parameter estimates is given in Figure 6.

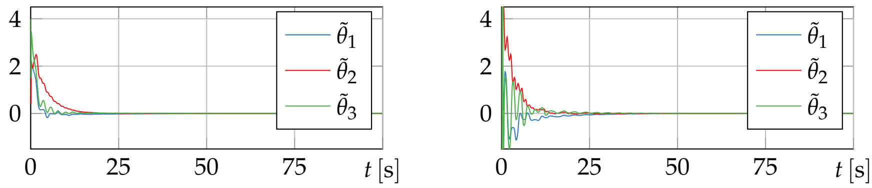

Figure 6. Results of the simulation with regressor depending on time only and different adaptation gains. (**Left**): $\boldsymbol{\Gamma} = 0.5\,\mathbf{I}$, (**Right**): $\boldsymbol{\Gamma} = 5\,\mathbf{I}$.

It can be seen that an increase of the adaptation gains enhances the adaptation speed at the cost of increasing the oscillations of the parameter estimates.

Now, the more complex problem of identification of parameters of the system with the regressor directly depending on unmeasurable state variables can be considered. For this scenario, a third-order system ($n = 3$) with three unknown parameters ($k = 3$) is chosen. The projection operator in the adaptation law is designed as in Remark 2 with a boundary θ_M chosen significantly greater than real values of the dynamic parameter in the system. This illustrates that only a limited a priori knowledge of the values of the parameters is sufficient for the effective application of the proposed PIESO algorithm. The settings of this simulation are presented in Table 3.

Table 3. Parameters of the simulation of PIESO with state dependent regressor.

ω_o	$\boldsymbol{\Gamma}$	$\boldsymbol{\psi}(t)$	$\boldsymbol{\theta}$	$\mathbf{x}(0)$	$\hat{\mathbf{z}}(0)$	$\hat{\boldsymbol{\theta}}(0)$	$bu(t)$	θ_M
250	$0.1\mathbf{I}$	$\mathbf{x}^T$	$\begin{bmatrix}-1\\-3\\-3\end{bmatrix}$	$\begin{bmatrix}1\\1\\1\end{bmatrix}$	$\mathbf{0}_{4\times1}$	$\mathbf{0}_{3\times1}$	$2\sin(t) + 2\sin(3t)$	100

To guarantee the boundedness of the state vector **x**, the parameters of the plant are chosen taking into account the asymptotic stability conditions of the process. The input u(t) is also changed, to guarantee the sufficient excitation of the regressor, which now indirectly depends on the input signal. The results of the simulation are given in Figure 7. The plot of auxiliary $V_z(\bar{\mathbf{z}})$ is omitted here, as during the trials it was observed that it serves its purposes only in some extreme cases of very large initial estimation errors. In all other cases, initial estimation errors seem to meet the inequality $\|\bar{\mathbf{z}}(0)\| \leq z_M$ and thus the function $V_{tz}(\bar{\mathbf{z}}, \tilde{\boldsymbol{\theta}})$ satisfies the conditions of Lyapunov function.

It can be seen that in the case of state-dependent regressor, a proper identification and estimation was also performed by the proposed PIESO algorithm. Estimation errors of all state variables and dynamic parameters converge to the origin. Again, vanishing of the total disturbance estimate is observed as the identified parameters approach their nominal values. Several drawbacks of the proposed scheme must also be admitted. Mainly, it can be seen that despite the small difference between the initial estimates $\hat{\boldsymbol{\theta}}(0)$ and the real values of $\boldsymbol{\theta}$, the estimates quickly reached values as high as θ_M only to start converging from this new configuration. This effect is caused by the peaking phenomenon of ESO in the presence of non-zero initial estimation errors and is the reason for the necessity of a projection operator. It could be mitigated in practice by disabling the adaptation in the first moments of work of the observer or by employing one of several modified ESO algorithms which are characterized by a limited presence of peaking phenomenon [61–63]. In a direct consequence of this phenomenon, the time of identification is significantly extended in comparison with the case with the regressor independent of the state of the system. Despite these flaws, the conducted simulations proved the efficiency of the proposed algorithm in estimating both the state and parameters with the asymptotic convergence of the estimation errors.

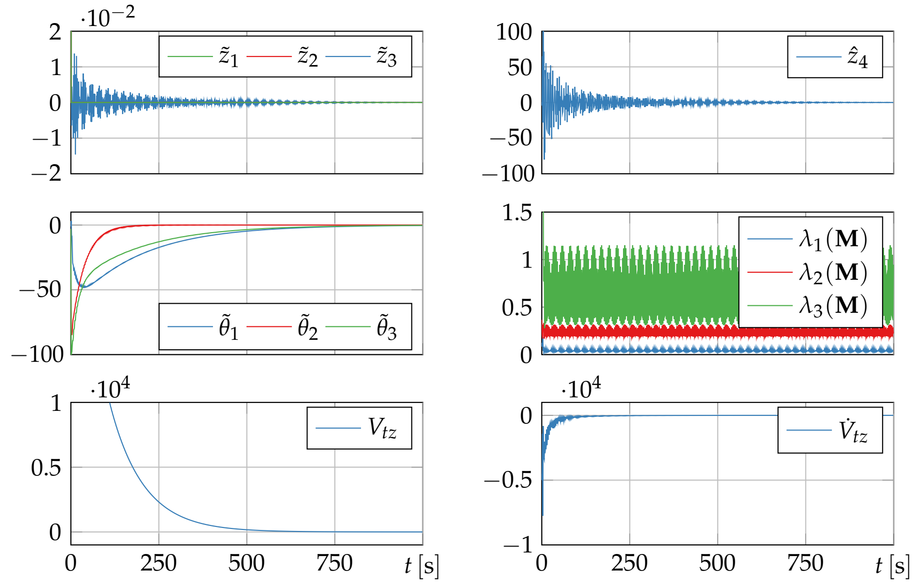

Figure 7. Results of the simulation with regressor depending on the state.

To further investigate the properties of the proposed algorithm, the simulation is repeated with different adaptation gains. As can be seen from the obtained parameter estimates presented in Figure 8, the speed of adaptation is increased and is here comparable with the convergence rate obtained for the case where the state independent regressor was studied, cf. Figure 5. Furthermore, in comparison with the results shown in Figure 7, the appearance of previously absent oscillations in estimates can be observed in Figure 8.

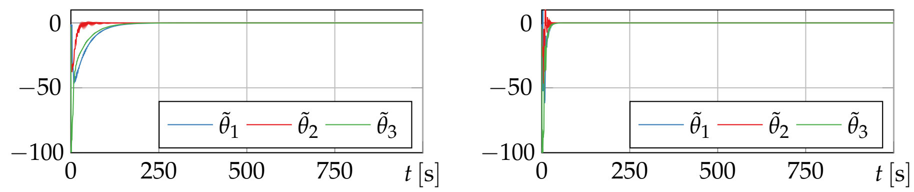

Figure 8. Results of the simulation with regressor depending on the state and different adaptation gains. (**Left**): $\mathbf{\Gamma} = 0.5\,\mathbf{I}$, (**Right**): $\mathbf{\Gamma} = 5\,\mathbf{I}$.

4.2. Harmonic Disturbance Frequency Identification

While the proposed algorithm is designed for the class of systems represented by (6) and satisfying Assumption 1, for which the PIESO can be easily applied, it can also be used to solve other problems studied in the literature that do not directly conform to the assumed nominal dynamics. As an example, the problem of identification of a frequency parameter of a harmonic disturbance is discussed in [51]. To this end, the use of Resonant ESO (RESO) [64–66] with an adaptive frequency estimator [1,67] was proposed to simultaneously estimate the disturbance itself and identify its frequency. By a slight modification of the RESO algorithm, one can employ the proposed PIESO to achieve the same goal. Consider the system in the form of

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}_n\mathbf{x} + \mathbf{b}_n f(t), \\ f(t) &= a_{\mathrm{r}}\sin(\omega_{\mathrm{r}} t), \end{aligned} \tag{54}$$

where $a_r, \omega_r \in \mathbb{R}_+$ are some unknown constant values denoting the amplitude and frequency of the harmonic disturbance. In [64], it is noticed that the disturbance $f(t)$ can be considered as a response of a harmonic oscillator with resonant frequency ω_r and dynamics given by

$$\ddot{f}_r + \omega_r^2 f_r = 0. \quad (55)$$

Thus, it is proposed to formulate the extended state for the system (54) and extend it with three additional states corresponding to f, $\dot{f}_r$, and $\ddot{f}_r$. In fact, one could extend the state $\mathbf{x}$ by f_r and $\dot{f}_r$ only, but for the sake of coherence with [51,64] the first option is considered here. Thus, the new state of the plant is rewritten as

$$\dot{\boldsymbol{\chi}} = \mathbf{A}_{n+3}\boldsymbol{\chi} - \mathbf{b}_{n+3}\omega_r^2\chi_{n+2}, \quad (56)$$

where $\boldsymbol{\chi} = \begin{bmatrix}\chi_1 & \dots & \chi_{n+3}\end{bmatrix}^T = \begin{bmatrix}\mathbf{x}^T & f & \dot{f}_r & \ddot{f}_r\end{bmatrix}^T$. It can be noticed that system (56) corresponds to the nominal system (6) with $b = 0, \theta = -\omega_r^2$ and $\boldsymbol{\psi}(\boldsymbol{\chi}) = \chi_{n+2} = \dot{f}_r$. Thus, the proposed identification scheme can be applied to this system by treating the state $\boldsymbol{\chi}$ as a nominal state of the plant, which can be further extended under the notion that $\delta = \dot{f}_r(-\omega_r^2 - \hat{\theta})$. To evaluate the performance of the proposed solution, a simulation was performed for a second-order system corresponding to (54). The parameters of this simulation are given in Table 4.

Table 4. Parameters of the simulation of PIESO used for frequency identification.

ω_o	Γ	$\boldsymbol{\psi}(t)$	θ	$\mathbf{x}(0)$	$\hat{\mathbf{z}}(0)$	$\hat{\theta}(0)$	θ_M
50	$0.01\mathbf{I}$	$\dot{f}_r$	-4	$\begin{bmatrix}0\\0\end{bmatrix}$	$\mathbf{0}_{6\times1}$	0	100

The results of the simulations are given in Figure 9, where the estimation errors of the state variables $\mathbf{x}$ and the harmonic disturbance f are shown, as well as the estimate of the total disturbance δ and the error of the parameter identification.

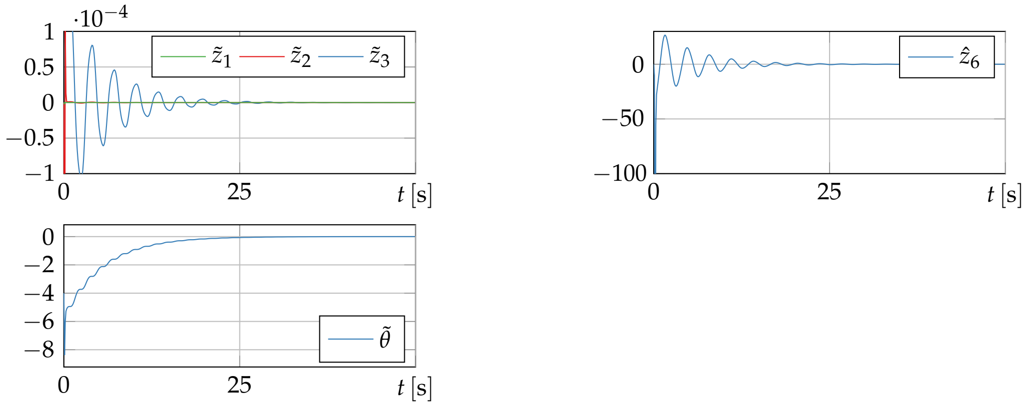

Figure 9. Results of the simulation with frequency identification.

It can be seen that the parameter identification error converges to the origin, which corresponds to the correct identification of unknown parameter $\theta = -4$ and thus the frequency of $\omega_r = 2$. Moreover, it can be noted that the estimation errors of state variables also converge and that the harmonic disturbance is properly estimated. Thus, the presented example shows that the proposed identification scheme can be successfully applied to some class of the systems nominally not conforming to Assumption 1.

5. Experimental Validation

Some experimental verification of the proposed algorithm has recently been presented in [68], where it was applied to identify a set of dynamic parameters of an underacturated hovercraft vehicle. To further evaluate the performance of the considered identification scheme, a new experimental trial is conducted here using a planar robotic manipulator with a single degree of freedom (PME1R). The employed PME1R system consists of a permanent magnet synchronous motor connected with a load through a reduction gear. The manipulator is controlled by a PC class computer taking advantage of VisSim software and a hardware I/O board. The hardware I/O board allows controlling of the manipulator through a single signal associated with the voltage on the motor and measuring of the momentary position and velocity of the motor shaft. All input and output signals are in form of analogue voltages in the range of $(-10, 10)$V and are later recomputed to obtain information about the factual state of the plant. The main control and acquisition loop operates at a frequency of 100 Hz. The photography of the considered manipulator is given in Figure 10.

Figure 10. Planar PME1R manipulator with a single degree of freedom.

While the PME1R manipulator is characterized by the flexible connection between the geared motor and the load itself and it is possible to measure positions of both the motor and the load independently, during this trial the flexibility present in the manipulator is treated as an unknown external disturbance and the plant itself is treated as a stiff one. The dynamics of the manipulator is modeled as a second order system ($n = 2$) in the form of

$$\dot{\mathbf{x}} = \mathbf{A}_n\mathbf{x} + \mathbf{b}_n(bu + dx_2 + q), \tag{57}$$

with unknown parameters to be identified being an input gain b, a friction coefficient d, and some constant offset coefficient q included to compensate for external disturbances, unmodeled dynamics, and actuator errors. As the regressor here depends on the state of the system, the identification algorithm of Section 3.2 was used with the projection operator as given in the Remark 2. The parameters of the employed identifier are given in Table 5.

Table 5. Parameters of the simulation of PIESO used for frequency identification.

ω_o	Γ	$\psi(t)$	θ	$\mathbf{x}(0)$	$\hat{\mathbf{z}}(0)$	$\hat{\theta}(0)$	θ_M
5	$\begin{bmatrix} 15 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} u \\ x_2 \\ 1 \end{bmatrix}^T$	$\begin{bmatrix} b \\ d \\ q \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	1000

The control signal u is chosen as a series of jump changes of input voltage designed to guarantee the boundedness of the state of the plant. The results of the experiment are given in Figure 11.

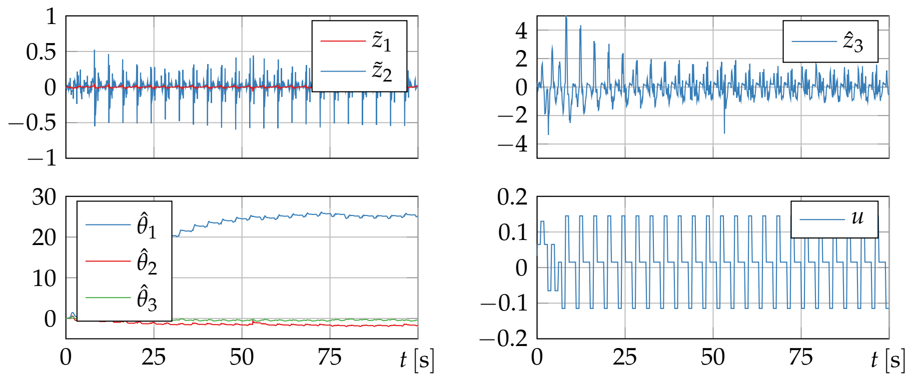

Figure 11. Results of the experiment.

On the graphs the estimation errors, the value of the total disturbance estimate, parameter estimates, and the control signal are given. It can be noticed that the PIESO efficiently tracks the real values of the state variables with errors of the position estimate being close to zero throughout the whole experiment. Although the error of velocity estimation is significantly higher, one should be aware that the quality of velocity measurement in the considered manipulator is also significantly worse than that of position measurement, and thus the presented $\tilde{z}_2$ error may not truly express the performance of the algorithm. It can also be observed that the estimates of all parameters converge to some constant values and the estimate of the total disturbance vanishes as this identification progresses, especially up until $t = 25$ s when the identification is most rapid. This is consistent with the results of theoretical analysis and simulations and allows one to infer that unknown parameters are identified correctly. The final values of the estimated parameters are

$$\hat{b} \approx 25.308, \hat{d} \approx -1.537, \hat{q} \approx -0.502. \tag{58}$$

To verify the obtained parameter estimates an additional experiment is performed and compared with the results of the simulation of model (57) using the parameters (58). The initial conditions of the simulation are chosen to match the momentary state of the experimental plant in a chosen time instant. In Figure 12, a comparison is given between the position and velocity from experiment and simulation.

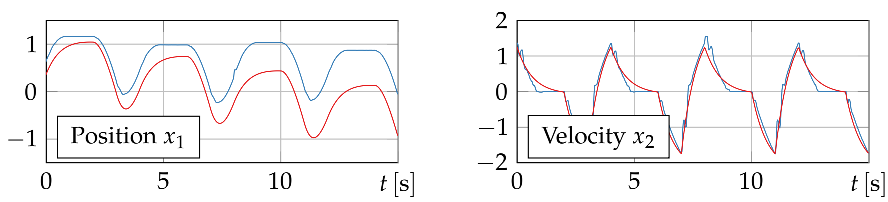

Figure 12. Comparison of the state trajectories obtained for the experimental system and the identified model. **Blue**: results of experiment, **red**: results of simulation.

Despite strong nonlinearities in the experimental plant, visible in the plot of the estimated velocity, the assumed linear model reproduces the behavior of the plant with a satisfying precision. The trajectories of the velocity obtained in both experiment and simulation bear a strong resemblance, and the trajectory of position differs in slight drift mainly. This drift could be caused by the imprecision of the measurement of the experimental plant or incorrect initial conditions set in the simulation.

6. Conclusions

In this paper, the novel adaptive observer is formulated in the framework of Active Disturbance Rejection Control. Algorithm of Parameter Identifying ESO is proposed for two distinct cases under the assumption of persistency of excitation. Firstly, the scenario of a fully known regressor is considered for which the PIESO consists of an adaptive ESO with gradient adaptation law. Secondly, the more complex case in which the regressor contains unmeasurable state of the plant is considered and PIESO is designed as adaptive ESO with the projected gradient identification law. It is formally proved that both algorithms guarantee the global asymptotic convergence of the estimation and identification errors to the origin for the correctly chosen gains. A simple tuning rule for the proposed method is given by stating that the observer bandwidth should be chosen high enough, while the identification gain should remain small enough in both scenarios. The proposed approach could be further improved by applying it to the task of full adaptive control in the ADRC framework. The experimental studies carried out show that the proposed estimation and identification method can be used to model and control actuators and electromechanical systems in robotics.

In addition to the main result of the paper, a new proof of stability of the classic gradient identification algorithm is given in the form of the strict Lyapunov function. This result is based on solutions recently proposed in the literature for other parameter identification methods.

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References

1. Sastry, S.; Bodson, M. *Adaptive Control: Stability, Convergence, and Robustness*; Prentice Hall: Englewood Cliff, NJ, USA, 1989.
2. Krstic, M.; Kokotovic, P.V.; Kanellakopoulos, I. *Nonlinear and Adaptive Control Design*; John Wiley & Sons, Inc.: New York, NY, USA, 1995.
3. Narendra, K.S.; Annaswamy, A.M. *Stable Adaptive Systems*; Prentice Hall: Englewood Cliff, NJ, USA, 1989.
4. Aranovskiy, S.; Bobtsov, A.; Ortega, R.; Pyrkin, A. Parameters estimation via dynamic regressor extension and mixing. In Proceedings of the 2016 American Control Conference (ACC), Boston, MA, USA, 6–8 July 2016; pp. 6971–6976.
5. Aranovskiy, S.; Belov, A.; Ortega, R.; Barabanov, N.; Bobtsov, A. Parameter identification of linear time-invariant systems using dynamic regressor extension and mixing. *Int. J. Adapt. Control. Signal Process.* **2019**, *33*, 1016–1030. [CrossRef]
6. Di Bernardo, M.; Montanaro, U.; Santini, S. Novel switched model reference adaptive control for continuous piecewise affine systems. In Proceedings of the 2008 47th IEEE Conference on Decision and Control, Cancun, Mexico, 9–11 December 2008; pp. 1925–1930. [CrossRef]
7. Siciliano, B.; Book, W.J. Model reference adaptive control of a link flexible arm. In Proceedings of the 1986 25th IEEE Conference on Decision and Control, Athens, Greece, 10–12 December 1986; pp. 91–95.
8. Guzman, E.; Moreno, J.A. A new finite-time convergent and robust direct model reference adaptive control for SISO linear time invariant systems. In Proceedings of the 2011 50th IEEE Conference on Decision and Control and European Control Conference, Orlando, FL, USA, 12–15 December 2011; pp. 7027–7032.

9. Hovakimyan, N.; Cao, C. *L1 Adaptive Control Theory: Guaranteed Robustness with Fast Adaptation*; SIAM: Philadelphia, PA, USA, 2010.
10. Jafari, S.; Ioannou, P.; Rudd, L.E. What is L1 adaptive control. In Proceedings of the AIAA Guidance, Navigation, and Control (GNC) Conference, Boston, MA, USA, 19–22 August 2013; p. 4513.
11. Luders, G.; Narendra, K. An adaptive observer and identifier for a linear system. *IEEE Trans. Autom. Control.* **1973**, *18*, 496–499. [CrossRef]
12. Carroll, R.; Lindorff, D. An adaptive observer for single-input single-output linear systems. *IEEE Trans. Autom. Control.* **1973**, *18*, 428–435. [CrossRef]
13. Afri, C.; Andrieu, V.; Bako, L.; Dufour, P. Identification of linear systems with nonlinear luenberger observers. In Proceedings of the 2015 American Control Conference (ACC), Chicago, IL, USA,1–3 July 2015; pp. 3373–3378.
14. Besancon, G.; de León-Morales, J.; Huerta-Guevara, O. On adaptive observers for state affine systems. *Int. J. Control.* **2006**, *79*, 581–591. [CrossRef]
15. Zhang, Q. Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems. *IEEE Trans. Autom. Control.* **2002**, *47*, 525–529. [CrossRef]
16. Postoyan, R.; Zhang, Q. Local adaptive observer for linear time-varying systems with parameter-dependent state matrices. In Proceedings of the 2018 IEEE Conference on Decision and Control (CDC), Miami Beach, FL, USA, 17–19 December 2018; pp. 4649–4654.
17. Bobtsov, A.; Ortega, R.; Yi, B.; Nikolaev, N. Adaptive state estimation of state-affine systems with unknown time-varying parameters. *Int. J. Control.* **2021**. [CrossRef]
18. Cho, Y.M.; Rajamani, R. A systematic approach to adaptive observer synthesis for nonlinear systems. *IEEE Trans. Autom. Control.* **1997**, *42*, 534–537.
19. Besancon, G.; Ţiclea, A. On adaptive observers for systems with state and parameter nonlinearities. *IFAC Pap.* **2017**, *50*, 15416–15421. [CrossRef]
20. Xing, Y.; Na, J.; Costa-Castelló, R.; Gao, G. Adaptive parameter estimation-based observer design for nonlinear systems. In Proceedings of the 2020 59th IEEE Conference on Decision and Control (CDC), Jeju Island, Korea, 14–18 December 2020; pp. 4170–4175.
21. Ortega, R.; Bobtsov, A.; Pyrkin, A.; Aranovskiy, S. A parameter estimation approach to state observation of nonlinear systems. *Syst. Control. Lett.* **2015**, *85*, 84–94. [CrossRef]
22. Åström, K.J.; Torsten, B. Numerical identification of linear dynamic systems from normal operating records. *IFAC Proc. Vol.* **1965**, *2*, 96–111. [CrossRef]
23. Narendra, K.S.; Annaswamy, A.M. Persistent excitation in adaptive systems. *Int. J. Control.* **1987**, *45*, 127–160. [CrossRef]
24. Panteley, E.; Loria, A.; Teel, A. Relaxed persistency of excitation for uniform asymptotic stability. *IEEE Trans. Autom. Control.* **2001**, *46*, 1874–1886. [CrossRef]
25. Lee, T.C. On the equivalence relations of detectability and PE conditions with applications to stability analysis of time-varying systems. In Proceedings of the 2003 American Control Conference, Denver, CO, USA, 4–6 June 2003; Volume 3, pp. 1873–1878.
26. Loría, A.; Panteley, E.; Teel, A. A new notion of persistency-of-excitation for UGAS of NLTV systems: Application to stabilisation of nonholonomic systems. In Proceedings of the 1999 European Control Conference (ECC), Karlsruhe, Germany, 31 August–3 September 1999; pp. 1363–1368.
27. Gibson, T.E.; Annaswamy, A.M. Adaptive control and the definition of exponential stability. In Proceedings of the 2015 American Control Conference (ACC), Chicago, IL, USA, 1–3 July 2015; pp. 1549–1554.
28. Loría, A.; Kelly, R.; Teel, A.R. Uniform parametric convergence in the adaptive control of mechanical systems. *Eur. J. Control.* **2005**, *11*, 87–100. [CrossRef]
29. Loria, A.; Panteley, E. Uniform exponential stability of linear time-varying systems: revisited. *Syst. Control. Lett.* **2002**, *47*, 13–24. [CrossRef]
30. Mazenc, F.; De Queiroz, M.; Malisoff, M. Uniform global asymptotic stability of a class of adaptively controlled nonlinear systems. *IEEE Trans. Autom. Control.* **2009**, *54*, 1152–1158. [CrossRef]
31. Loria, A.; Panteley, E.; Maghenem, M. Strict Lyapunov functions for model-reference adaptive control based on the Mazenc construction. In Proceedings of the Congreso Nacional de Control Automático (CNCA 2019). Asociación de México de Control Automático, Puebla, Mexico, 23–25 October 2019; pp. 407–412.
32. Gao, Z. Active disturbance rejection control: A paradigm shift in feedback control system design. In Proceedings of the 2006 American Control Conference, Minneapolis, MN, USA, 14–16 June 2006; pp. 2399–2405.
33. Gao, Z.; Huang, Y.; Han, J. An alternative paradigm for control system design. In Proceedings of the 40th IEEE Conference on Decision and Control, Orlando, FL, USA, 4–7 December 2001; Volume 5, pp. 4578–4585.
34. Han, J. From PID to Active Disturbance Rejection Control. *IEEE Trans. Ind. Electron.* **2009**, *56*, 900–906. [CrossRef]
35. Qi, X.; Li, J.; Xia, Y.; Gao, Z. On the robust stability of active disturbance rejection control for SISO systems. *Circuits, Syst. Signal Process.* **2017**, *36*, 65–81. [CrossRef]
36. Xue, W.; Huang, Y. On performance analysis of ADRC for a class of MIMO lower-triangular nonlinear uncertain systems. *ISA Trans.* **2014**, *53*, 955–962. [CrossRef]

37. Chen, Z.; Wang, Y.; Sun, M.; Sun, Q. Convergence and stability analysis of active disturbance rejection control for first-order nonlinear dynamic systems. *Trans. Inst. Meas. Control.* **2019**, *41*, 2064–2076. [CrossRef]
38. Xue, W.; Huang, Y. On performance analysis of ADRC for nonlinear uncertain systems with unknown dynamics and discontinuous disturbances. In Proceedings of the 32nd Chinese Control Conference, Xi'an, China, 26–28 July 2013; pp. 1102–1107.
39. Patelski, R.; Dutkiewicz, P. On the stability of ADRC for manipulators with modelling uncertainties. *ISA Trans.* **2020**, *102*, 295–303. [CrossRef]
40. Wang, Y.; Liu, J.; Chen, Z.; Sun, M.; Sun, Q. On the stability and convergence rate analysis for the nonlinear uncertain systems based upon active disturbance rejection control. *Int. J. Robust Nonlinear Control.* **2020**, *30*, 5728–5750. [CrossRef]
41. Zheng, Q. On Active Disturbance Rejection Control: Stability Analysis and Applications in Disturbance Decoupling Control. Ph.D. Thesis, Cleveland State University, Cleveland, OH, USA, 2009.
42. Yao, J.; Jiao, Z.; Ma, D. Adaptive robust control of DC motors with extended state observer. *IEEE Trans. Ind. Electron.* **2013**, *61*, 3630–3637. [CrossRef]
43. Deng, W.; Yao, J. Extended-state-observer-based adaptive control of electrohydraulic servomechanisms without velocity measurement. *IEEE/ASME Trans. Mechatron.* **2019**, *25*, 1151–1161. [CrossRef]
44. Madoński, R.; Gao, Z.; Łakomy, K. Towards a turnkey solution of industrial control under the active disturbance rejection paradigm. In Proceedings of the 2015 54th Annual Conference of the Society of Instrument and Control Engineers of Japan (SICE), Hangzhou, China, 28–30 July 2015; pp. 616–621.
45. Madoński, R. On Active Disturbance Rejection in Robotic Motion Control. Ph.D. Thesis, Poznan University of Technology, Poznan, Poland, 2016.
46. Huang, C.; Guo, L. Control of a class of nonlinear uncertain systems by combining state observers and parameter estimators. In Proceedings of the 10th World Congress on Intelligent Control and Automation, Beijing, China, 6–8 July 2012; pp. 2054–2059. [CrossRef]
47. Wang, Y.; Chen, Z.; Sun, M.; Sun, Q.; Piao, M. On Sign Projected Gradient Flow Optimized Extended State Observer Design for a Class of Systems with Uncertain Control Gain. *IEEE Trans. Ind. Electron.* **2022**, doi: [CrossRef]
48. Jiang, T.; Huang, C.; Guo, L. Control of uncertain nonlinear systems based on observers and estimators. *Automatica* **2015**, *59*, 35–47. [CrossRef]
49. Lozada-Castillo, N.; Luviano-Juárez, A.; Chairez, I. Robust control of uncertain feedback linearizable systems based on adaptive disturbance estimation. *ISA Trans.* **2019**, *87*, 1–9. [CrossRef]
50. Palomino-Resendiz, S.I.; Lozada-Castillo, N.B.; Flores-Hernández, D.A.; Gutiérrez-Frías, O.O.; Luviano-Juárez, A. Adaptive Active Disturbance Rejection Control of Solar Tracking Systems with Partially Known Model. *Mathematics* **2021**, *9*, 2871. [CrossRef]
51. Madonski, R.; Stanković, M.; Shao, S.; Gao, Z.; Yang, J.; Li, S. Active disturbance rejection control of torsional plant with unknown frequency harmonic disturbance. *Control. Eng. Pract.* **2020**, *100*, 104413. [CrossRef]
52. Yang, X.; Huang, Y. Capabilities of extended state observer for estimating uncertainties. In Proceedings of the 2009 American Control Conference, Saint Louis, MO, USA, 10–12 June 2009; pp. 3700–3705.
53. Guo, B.Z.; Zhao, Z.L. On the convergence of an extended state observer for nonlinear systems with uncertainty. *Syst. Control. Lett.* **2011**, *60*, 420–430. [CrossRef]
54. Madoński, R.; Herman, P. Survey on methods of increasing the efficiency of extended state disturbance observers. *ISA Trans.* **2015**, *56*, 18–27. [CrossRef]
55. Lion, P.M. Rapid identification of linear and nonlinear systems. *AIAA J.* **1967**, *5*, 1835–1842. [CrossRef]
56. Morgan, A.; Narendra, K. On the uniform asymptotic stability of certain linear nonautonomous differential equations. *SIAM J. Control. Optim.* **1977**, *15*, 5–24. [CrossRef]
57. Anderson, B. Exponential stability of linear equations arising in adaptive identification. *IEEE Trans. Autom. Control.* **1977**, *22*, 83–88. [CrossRef]
58. Gao, Z. Scaling and bandwidth-parameterization based controller tuning. In Proceedings of the American Control Conference, Minneapolis, MN, USA, 14–16 June 2006; Volume 6, pp. 4989–4996.
59. Lavretsky, E.; Gibson, T.E.; Annaswamy, A.M. Projection Operator in Adaptive Systems. *arXiv* **2012**, arXiv:nlin.AO/1112.4232.
60. Yao, B.; Tomizuka, M. Smooth robust adaptive sliding mode control of manipulators with guaranteed transient performance. In Proceedings of the 1994 American Control Conference—ACC '94, Baltimore, MD, USA, 29 June–1 July 1994; Volume 1, pp. 1176–1180. [CrossRef]
61. Astolfi, D.; Marconi, L.; Praly, L.; Teel, A.R. Low-power peaking-free high-gain observers. *Automatica* **2018**, *98*, 169–179. [CrossRef]
62. Barreiro, A.; Delgado, E. Reset observers alleviating the peaking and the robustness tradeoffs: A case study on force estimation in teleoperation. *ISA Trans.* **2019**, *94*, 36–46. [CrossRef] [PubMed]
63. Kasac, J.; Pender, A.; Pranjic, M.; Kotarski, D. Frequency-shifting-based algebraic approach to extended state observer design. *Asian J. Control.* **2021**, *23*, 2171–2184. [CrossRef]
64. Stanković, M.R.; Rapaić, M.R.; Manojlović, S.M.; Mitrović, S.T.; Simić, S.M.; Naumović, M.B. Optimised active disturbance rejection motion control with resonant extended state observer. *Int. J. Control.* **2019**, *92*, 1815–1826. [CrossRef]
65. Stanković, M.R.; Madonski, R.; Shao, S.; Mikluc, D. On dealing with harmonic uncertainties in the class of active disturbance rejection controllers. *Int. J. Control.* **2021**, *94*, 2795–2810. [CrossRef]

66. Łakomy, K.; Patelski, R.; Pazderski, D. ESO architectures in the trajectory tracking ADR controller for a mechanical system: A comparison. In *Advanced, Contemporary Control, Proceedings of the KKA 2020—The 20th Polish Control Conference, Lodz, Poland, 14–29 October 2020*; Springer: Berlin/Heidelberg, Germany, 2020; pp. 1323–1335.
67. Hou, M. Estimation of sinusoidal frequencies and amplitudes using adaptive identifier and observer. *IEEE Trans. Autom. Control.* **2007**, *52*, 493–499. [CrossRef]
68. Patelski, R.; Pazderski, D. Extended State Observer Based Parameter Identification of the Hovercraft System. In Proceedings of the 26th International Conference on Methods and Models in Automation and Robotics, Miedzyzdroje, Poland, 22–25 August 2022. (*in review*).